American Digital Call Option

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The American digital call has fixed payoff of 1$ if the barrier is being triggered before maturity and zero otherwise. It follows that its price has to satisfy $V_t \leq 1$. Furthermore, if $S_t \geq K$, the immediate exercise of the call would pay the owner the current intrinsic value of 1$ and thus its price has to satisfy $V_t \geq 1$ in this situation. Combining these two arguments, we see that the price of the American call has to be equal to the intrinsic value of 1$ if $S_t \geq K$. Since it is optimal to exercise American derivative securities, when their price is equal to the intrinsic value, it follows that it is optimal to exercise the American digital call at the first time that the level K is reached.

We first need to derive the first passage time density of a drifted Brownian motion to an upper level $m > 0$. From Theorem 3.7.1 in Steven Shreve's book "Stochastic Calculus for Finance II", we know that the first passage time of a non-drifted Brownian motion to this level has the distribution function and density

$$\mathbb{P}\left\{\tau_m \leq t\right\} = \frac{2}{2\pi}\int_{\frac{m}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}}dy, \quad f_{\tau_m}(t) = \frac{m}{t\sqrt{2\pi t}}e^{-\frac{m^2}{2t}}$$

Let $\tilde{W}(t)$ be a non-drifted Brownian motion under the risk-neutral measure $\tilde{\mathbb{P}}$. We now define a new Brownian motion $\hat{W}(t)$ by

$$\hat{W}(t) = \tilde{W}(t) + \theta t, \qquad \theta \in \mathbb{R}$$

Note that $\hat{W}(t)$ has a non-zero drift under the risk-neutral measure. Define the Radon-Nikodým derivative process $Z(t)$ by

$$Z(t) = \exp\left\{-\theta\tilde{W}(t) - \frac{1}{2}\theta^2 t\right\}$$

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By Girsanov's Theorem, $\hat{W}(t)$ is a Brownian motion under the probability measure $\hat{\mathbb{P}}$ defined by

$$\hat{\mathbb{P}}(A) = \int_A Z(t) d\tilde{\mathbb{P}} \qquad \forall A \in \Omega$$

It thus follows that

$$\begin{aligned}
\mathbb{P}\left\{\tau_m \leq t\right\} &= \tilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau_m \leq t\}}\right] \\
&= \hat{\mathbb{E}}\left[\frac{1}{Z(T)}\mathbb{I}_{\{\tau_m \leq t\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left\{\theta \hat{W}(T) - \frac{1}{2}\theta^2 T\right\}\mathbb{I}_{\{\tau_m \leq t\}}\right] \\
&= \hat{\mathbb{E}}\left[\mathbb{E}\left[\exp\left\{\theta \hat{W}(T) - \frac{1}{2}\theta^2 T\right\}\middle| \mathcal{F}\left(\tau_m \wedge t\right)\right]\mathbb{I}_{\{\tau_m \leq t\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left\{\theta \hat{W}\left(\tau_m \wedge t\right) - \frac{1}{2}\theta^2\left(\tau_m \wedge t\right)\right\}\mathbb{I}_{\{\tau_m \leq t\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left\{\theta m - \frac{1}{2}\theta^2 \tau_m\right\}\mathbb{I}_{\{\tau_m \leq t\}}\right] \\
&= \int_0^t \frac{m}{s\sqrt{2\pi s}} \exp\left\{-\frac{m^2 - 2m\theta s - \theta^2 s^2}{2s}\right\} ds \\
&= \int_0^t \frac{m}{s\sqrt{2\pi s}} \exp\left\{-\frac{\left(m - \theta s\right)^2}{2s}\right\} ds
\end{aligned}$$

Taking the first derivative w.r.t. t finally yields the first passage time density of $\hat{W}(t)$ to the level m under $\tilde{\mathbb{P}}$

$$\hat{f}_{\tau_m}(t) = \frac{m}{t\sqrt{2\pi t}} \exp\left\{-\frac{\left(m - \theta t\right)^2}{2t}\right\}$$

In order to determine the level m, we first set

$$\theta = \frac{r - \frac{1}{2}\sigma^2}{\sigma}.$$

The solution to the geometric Brownian motion SDE is then given by

$$S(t) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma \tilde{W}(t)\right\} = S(0)\exp\left\{\sigma \hat{W}(t)\right\}$$

The spot price at time t triggers the barrier if

$$S(t) \geq B \quad \Leftrightarrow \quad \hat{W}(t) \geq \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma} = m$$

Under the risk-neutral measure, the current price of the American digital call is the expected discount factor of the first passage time to the barrier level m conditional on the barrier being triggered before the maturity. We get

$$
\begin{aligned}
V(0) &= \tilde{\mathbb{E}}\left[e^{-r\tau_m}\mathbb{I}_{\{\tau_m \leq T\}}\right] \\
&= \int_0^T e^{-rt}\frac{m}{t\sqrt{2\pi t}}\exp\left\{-\frac{(m-\theta t)^2}{2t}\right\}dt
\end{aligned}
$$

Now let $\alpha = \sigma\theta = r - \frac{1}{2}\sigma^2$. We substitute for m to get

$$
\begin{aligned}
\ldots &= \int_0^T e^{-rt}\frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right)-\alpha t\right)^2}{2\sigma^2 t}\right\}dt \\
&= \int_0^T e^{-rt}\frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\ln^2\left(\frac{B}{S(0)}\right)-2\ln\left(\frac{B}{S(0)}\right)\alpha t+\alpha^2t^2}{2\sigma^2 t}\right\}dt \\
&= \left(\frac{B}{S_0}\right)^{\frac{\alpha}{\sigma^2}}\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\ln^2\left(\frac{B}{S(0)}\right)+\left(\alpha^2+2\sigma^2 r\right)t^2}{2\sigma^2 t}\right\}dt
\end{aligned}
$$

Now let $\beta = \sqrt{\alpha^2+2\sigma^2 r}$, then

$$
\begin{aligned}
\ldots &= \left(\frac{B}{S_0}\right)^{\frac{\alpha}{\sigma^2}}\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\ln^2\left(\frac{B}{S(0)}\right)+\beta^2t^2}{2\sigma^2 t}\right\}dt \\
\ldots &= \left(\frac{B}{S_0}\right)^{\frac{\alpha}{\sigma^2}}\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\ln^2\left(\frac{B}{S(0)}\right)\pm 2\ln\left(\frac{B}{S(0)}\right)\beta t+\beta^2t^2}{2\sigma^2 t}\right\}dt \\
&= \left(\frac{B}{S_0}\right)^{\frac{\alpha\pm\beta}{\sigma^2}}\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right)\pm\beta t\right)^2}{2\sigma^2 t}\right\}dt
\end{aligned}
$$

We note that

$$
\frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma\sqrt{t}} = \frac{\ln\left(\frac{B}{S(0)}\right)-\beta t}{2\sigma\sqrt{t}}+\frac{\ln\left(\frac{B}{S(0)}\right)+\beta t}{2\sigma\sqrt{t}} \tag{1}
$$

and can thus split up the integral into

$$\begin{aligned} \cdots \quad = \quad & \left(\frac{B}{S_0}\right)^{\frac{\alpha\pm\beta}{\sigma^2}} \left[\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right) - \beta t}{2\sigma t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right) \pm \beta t\right)^2}{2\sigma^2 t}\right\} dt \right. \\ & \left. + \int_0^T \frac{\ln\left(\frac{B}{S(0)}\right) + \beta t}{2\sigma t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right) \pm \beta t\right)^2}{2\sigma^2 t}\right\} dt\right] \end{aligned}$$

We now evaluate these integral separately. For both integrals, we apply a change of variable from t to $x_\pm$ by setting

$$x_\pm(t) = \frac{\ln\left(\frac{B}{S(0)}\right) \pm \beta t}{\sigma\sqrt{t}}, \qquad \frac{dx_\pm}{dt} = -\frac{\ln\left(\frac{B}{S(0)}\right) \mp \beta t}{2\sigma t\sqrt{t}}$$

This suggests, to use the positive version of the first integral and change the variable to x_+ and the negative version of the second integral in combination with change of variable to x_-. This approach eliminates all terms depending on t inside the integrals. We further note that $\lim_{x_\pm \downarrow 0} = \infty$. The first integral becomes

$$\begin{aligned} & \left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \int_0^T \frac{\ln\left(\frac{B}{S(0)}\right) - \beta t}{2\sigma t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right) + \beta t\right)^2}{2\sigma^2 t}\right\} dt \\ & -\left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \int_\infty^{x_+(T)} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x_+^2}{2}\right\} dx_+ \\ = \quad & \left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \mathcal{N}\left(-x_+(T)\right) \\ = \quad & \left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \mathcal{N}\left(\frac{\ln\left(\frac{S(0)}{B}\right) - \beta T}{\sigma\sqrt{T}}\right) \end{aligned}$$

The second integral becomes

$$
\begin{aligned}
& \left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)+\beta t}{2\sigma t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right)-\beta t\right)^2}{2\sigma^2 t}\right\} dt \\
& -\left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \int_\infty^{x_+(T)} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x_-^2}{2}\right\} dx_- \\
= & \left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \mathcal{N}\left(-x_-(T)\right) \\
= & \left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \mathcal{N}\left(\frac{\ln\left(\frac{S(0)}{B}\right)+\beta T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

Thus, the value of the American digital call is

$$
V(0)=\left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \mathcal{N}\left(\frac{\ln\left(\frac{S(0)}{B}\right)-\beta T}{\sigma\sqrt{T}}\right)+\left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \mathcal{N}\left(\frac{\ln\left(\frac{S(0)}{B}\right)+\beta T}{\sigma\sqrt{T}}\right)
$$