The Cox-Ingersoll-Ross Model

Matthias Thul, Ally Quan Zhang

June 2, 2010

References

- Cox, John C.; Ingersoll, Jonathan E.; Ross, Stephen A.
  "An Intertemporal General Equilibrium Model of Asset Prices"
  Econometrica, Vol. 53, No. 2 (March 1985), pp. 363-384
- Cox, John C.; Ingersoll, Jonathan E.; Ross, Stephen A.
  "A Theory of the Term Structure of Interest Rates"
  Econometrica, Vol. 53, No. 2 (March 1985), pp. 385-407

Motivation

- develop a general framework to model the term structure of interest rates, price bonds and derivative products
- general decision between the
  (i) *arbitrage approach* with exogenously given interest rate dynamics and the
  (ii) *equilibrium approach* that determines them endogenously
- Cox-Ingersoll-Ross (CIR) adopt an equilibrium approach to endogenously determine the risk-free rate

Outline

I) outline the CIR general production economy framework

II) introduce a one-factor representation of the model economy

III) determine the optimal consumption strategy in the one-factor model

IV) derive the equilibrium risk-free rate

V) develop the dynamics of the risk-free rate

VI) price contingent claims in the one-factor model

VII) compare the equilibrium and the arbitrage approach

I) The general CIR Production Economy

-

Model Assumptions

1) **“consumption good”:**
   - single consumption good that cannot be stored and has to be either consumed or invested
   - serves as both, the input and the output of the production process
   - all values are measure in units of this good

2) **“production opportunities”:**

- n different risky technologies
- transformation process of production opportunity i is

$$\frac{d\eta_i(\mathbf{x},t)}{\eta_i(\mathbf{x},t)} = \mu_i(\mathbf{x},t)dt + \sigma_i(\mathbf{x},t)dz_i \qquad i = 1,\ldots,n$$

  where μ_i and σ_i are the exogenous instantaneous drift and diffusion, $\mathbf{x}$ is the vector of state variables and dz_i is the increment of a Wiener process
- the single good is both the input and the output of the production process
- assume that μ_i and σ_i fulfil conditions s.t. the above SDE is well-defined and has a unique solution
- *“constant returns to scale”*: the yield is independent of the invested volume due to the linearity of the SDE
- covariance $(\sigma_i dz_i)(\sigma_j dz_j) = \sigma_{i,j} dt$
- there are no limitations regarding the amount that can be invested into the production

3) **"state variables":**

- k factors representing the state of the technology
- the state variable i follows the process

$$dx_i = a_i(\mathbf{x}, t)dt + b_i(\mathbf{x}, t)d\zeta_i \qquad i = 1, \ldots, k$$

  where a_i and b_i are the local drift and diffusion functions
- covariance $(b_i d\zeta_i)(b_j d\zeta_j) = b_{i,j}dt$, $(\sigma_i dz_i)(b_j d\zeta_j) = \phi_{i,j}dt$
- the state vector $\mathbf{x}$ has to represent all necessary information in aggregate form

4) **"market":**

- continuous trading in frictionless market at equilibrium prices
- market participants are price-takers

5) **“individuals”:**

- identical individuals w.r.t. initial capital endowment, preferences and expectations
- the stochastic dynamics and current state of the economy are public knowledge
- at each point in time t, choose their instantaneous consumption C_t and portfolio weights ω_i, $i = 1, \ldots, n$ for the n production opportunities s.t. $\sum_{i=1}^{n} \omega_i = 1$ and $\omega_i \geq 0$
- maximise expected “lifetime utility” subject to the budget constraint

$$\max_{C_s, \omega_{i,s}, \forall s, i} \left\{ \mathbb{E}\left[\int_t^T U(C_s, s) ds \,\middle|\, \mathcal{F}_t \right] \right\}$$

where T is their planning horizon

6) **“utility functions”:**

- strictly increasing, concave, twice differentiable von Neumann-Morgenstern utility function with constant relative risk-aversion (RRA)

$$U(C_t, t) = e^{-\rho t}\left[\frac{C_t^{\gamma} - 1}{\gamma}\right]$$

where γ is one minus the coefficient of RRA

$$1 - \gamma := -C\frac{U_{CC}}{U_C} > 0$$

- $U(C_t, t)$ is an *“isoelastic utility function”*
- *“time separability”*: time preferences enter the utility only via the pre-factor $e^{-\rho t}$ where $\rho > 0$ is the time preference factor
- the relative proportion of risky investment from total wealth W is constant

7) **“risk-free rate”:**

- the instantaneous risk-free rate r_t is endogenously determined in equilibrium
- applies to all individuals and for both, borrowing and lending
- “shadow” riskless rate at which individuals are indifferent between borrowing and lending and choose zero transaction volume $\Rightarrow$ zero net supply in the whole economy

8) **“contingent claims”:**

- there exists a market for derivative instruments that have payoffs in units of the consumption good (e.g. bonds, futures, options)
- derivatives are in zero net supply
- the value $P(W_t, \mathbf{x}, t)$ could be a function of the aggregate wealth, the state vector and time
- equilibrium prices are independent of aggregate wealth due to the assumption of a constant RRA utility function (except if the payoff is a function of W_t)

Representative Individual

- the homogeneity assumptions allow to apply Rubinstein's aggregation theorem
- equilibrium prices can be determined assuming a representative individual (RI) who maximises expected utility under his budget constraint
- the RI invests the fraction of wealth that he does not consume into production due to zero net supply of the risk-free instrument and the contingent claims
- the dynamic budget equation is

$$dW_t = W_t \sum_{i}^{n} \omega_i \mu_i dt - C_t dt + W_t \sum_{i}^{n} \omega_i \sigma_i dz_i$$

II) A One-Factor Model Economy

- one production opportunity ($n = 1$) and one state variable ($k = 1$)

$$\frac{d\eta(x_t, t)}{\eta(x_t, t)} = \mu(x_t, t)dt + \sigma(x_t, t)dz_t$$

$$dx_t = a(x_t, t)dt + b(x_t, t)d\zeta_t$$

where $(\sigma dz_t)(bd\zeta_t) = \phi dt$

- the RI's optimization problem becomes

$$\max_{C_s, \forall s} \left\{ \mathbb{E} \left[\int_t^T U(C_s, s)ds \middle| \mathcal{F}_t \right] \right\}$$

subject to the dynamic budget equation

$$dW_t = (W_t \mu(x_t, t) - C_t)dt + W_t \sigma(x_t, t)dz_t$$

III) Optimal Consumption Strategy in the One-Factor Model

- define the indirect utility function $J(W_t, x_t, t)$ by

$$J(W_t, x_t, t) := \max_{C_s, \forall s} \left\{ \mathbb{E} \left[\int_t^T U(C_s, s) ds \middle| \mathcal{F}_t \right] \right\}$$

- according to Bellman's *Principle of Optimality*, optimal strategies are time consistent

$$\begin{aligned} J(W_t, x_t, t) &= \max_{C_s, \forall s} \left\{ \mathbb{E} \left[\int_t^{t+dt} U(C_s, s) ds \right.\right. \\ &\quad \left.\left. + \max_{C_s, \forall s} \left\{ \mathbb{E} \left[\int_{t+dt}^T U(C_s, s) ds \middle| \mathcal{F}_{t+dt} \right] \right\} \middle| \mathcal{F}_t \right] \right\} \\ &= \max_{C_s, \forall s} \left\{ \mathbb{E} \left[\int_t^{t+dt} U(C_s, s) ds + J(W_{t+dt}, x_{t+dt}, t + dt) \middle| \mathcal{F}_t \right] \right\} \end{aligned}$$

- assuming that we can apply Itô's lemma, $J(W_{t+dt}, x_{t+dt}, t+dt)$ can computed as

$$
\begin{aligned}
J(W_{t+dt}, x_{t+dt}, t+dt) &\approx J(W_t, x_t, t) + \frac{\partial J}{\partial W_t} dW_t + \frac{\partial J}{\partial x_t} dx_t + \frac{\partial J}{\partial t} dt \\
&+ \frac{1}{2}\frac{\partial^2 J}{\partial W_t^2} (dW_t)^2 + \frac{1}{2}\frac{\partial^2 J}{\partial x_t^2} (dx_t)^2 + \frac{\partial^2 J}{\partial W_t \partial x_t} dW_t dx_t \\
&= J(W_t, x_t, t) + \frac{\partial J}{\partial W_t}(W_t \mu - C_t) dt + \frac{\partial J}{\partial W_t} W_t \sigma dz_t \\
&+ \frac{\partial J}{\partial x_t} a dt + \frac{\partial J}{\partial x_t} b d\zeta_t + \frac{\partial J}{\partial t} dt + \frac{1}{2}\frac{\partial^2 J}{\partial W_t^2} W_t^2 \sigma^2 dt \\
&+ \frac{1}{2}\frac{\partial^2 J}{\partial x_t^2} b^2 dt + \frac{\partial^2 J}{\partial W_t \partial x_t} W_t \phi dt
\end{aligned}
$$

- note that the Wiener processes z and ζ are martingales and thus their changes have zero expected value

- plugging this result back into the definition of the indirect utility function results the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{aligned} 0 &= \max_{C_t} \left\{ U(C_t, t) + \frac{\partial J}{\partial W_t}(W_t \mu - C_t) + \frac{\partial J}{\partial x_t} a + \frac{\partial J}{\partial t} \right. \\ & \left. + \frac{1}{2} \frac{\partial^2 J}{\partial W_t^2} W_t^2 \sigma^2 + \frac{1}{2} \frac{\partial^2 J}{\partial x_t^2} b^2 + \frac{\partial^2 J}{\partial W_t \partial x_t} W_t \phi \right\} \end{aligned}$$

where $J(W, x, t)$ and dt were cancelled out

- using the Dynkin operator $\mathcal{L}$ to simplify the notation yields

$$0 = \max_{C_t} \left\{ U(C_t, t) + \mathcal{L} J(W_t, x_t, t) \right\}$$

- the optimization problem can be solved in three steps:
  1) determine the optimal consumption level $C_t^*(J)$ depending on the indirect utility function J
  2) recover J by solving the PDE that is obtained by substituting the optimal consumption $C_t^*(J)$ into the Bellman equation
  3) solve for the optimal consumption level C_t^* by substituting the indirect utility function into $C_t^*(J)$

Step 1: Determine $\mathbf{C_t^*(J)}$

- let $0 = \max_{C_t} \{\mathbf{\Psi}(C_t)\}$ s.t. $C_t \geq 0$, then

$$\begin{aligned}\mathbf{\Psi}(C_t) &= U(C_t, t) + \frac{\partial J}{\partial W_t}(W_t\mu - C_t) + \frac{\partial J}{\partial x_t}a + \frac{\partial J}{\partial t} \\ &\quad + \frac{1}{2}\frac{\partial^2 J}{\partial W_t^2}W_t^2\sigma^2 + \frac{1}{2}\frac{\partial^2 J}{\partial x_t^2}b^2 + \frac{\partial^2 J}{\partial W_t \partial x_t}W_t\phi\end{aligned}$$

- the Kuhn-Tucker first order conditions (FOC) for C_t are

$$\begin{aligned}\frac{\partial \mathbf{\Psi}}{\partial C_t} &= \frac{\partial U}{\partial C_t} - \frac{\partial J}{\partial W_t} \leq 0 \\ C_t\frac{\partial \mathbf{\Psi}}{\partial C_t} &= C_t\frac{\partial U}{\partial C_t} - C_t\frac{\partial J}{\partial W_t} = 0\end{aligned}$$

- except for the trivial solution $C_t = 0$, the optimal consumption is chosen such that marginal utility of consumption equals to the marginal utility of future wealth

$$\frac{\partial U}{\partial C_t} = \frac{\partial J}{\partial W_t}$$

- the FOC are not only necessary but also sufficient due to the strict concavity of the utility function U
- given the concrete utility function, the optimal consumption rate $C^*(J)$ can be determined

$$U(C_t, t) = e^{-\rho t}\left[\frac{C_t^{\gamma} - 1}{\gamma}\right] \qquad \Rightarrow \qquad \frac{\partial U}{\partial C_t} = e^{-\rho t}\left(C_t^*\right)^{\gamma - 1}$$

$$\Rightarrow \qquad C_t^*(J) = \left(e^{\rho t}\frac{\partial J}{\partial W_t}\right)^{\frac{1}{\gamma - 1}}$$

Step 2: Solve for J

- substituting back into the HJB equation and grouping similar terms yields a non linear PDE for J that can in general not be solved explicitly
- for isolastic utility functions, the indirect utility function takes the form

$$J(W_t, x_t, t) = f(x_t, t)U(W_t, t) + g(x_t, t)$$

- logarithmic utility is a limiting case of the isoelastic utility when $\gamma \to 0$ or equivalently RRA $\to 1$

$$\lim_{\gamma \to 0} e^{-\rho t} \left[\frac{C_t^\gamma - 1}{\gamma} \right] = e^{-\rho t} \ln(C_t)$$

- in this special case, $f(x, t)$ is independent of wealth W_t a function of time t only

$$f(x_t, t) = f(t) = \frac{1 - e^{-\rho(T-t)}}{\rho}$$

Step 3: Solve for $\mathbf{C_t^*}$

- differentiating $J(W_t, x_t, t)$ w.r.t. W_t yields

$$\frac{\partial}{\partial W_t}\left[f(t)U(W_t,t) + g(x_t,t)\right] = f(t)\frac{\partial U}{\partial W_t} = \frac{1 - e^{-\rho(T-t)}}{\rho W_t}$$

- using the FOC allows to solve for C_t^*

$$\frac{\partial U}{\partial C_t^*} = \frac{\partial J}{\partial W_t} \quad \Leftrightarrow \quad C_t^* = \frac{W_t \rho}{1 - e^{-\rho(T-t)}}$$

- under logarithmic utility C_t^* depends on the current wealth W_t, the time preference factor ρ and the planning horizon $(T-t)$ only
- the optimal consumption C_t^* is independent of the function $g(x_t, t)$ and thus of the production opportunities in the economy

IV) Equilibrium Risk-free Rate in the One-Factor Model

- since the risk-free instrument is in zero net supply, it is not being held by the representative investor
- determine the risk-free rate endogenously such that the investor is not better off by trading in the money market, i.e. he is indifferent between an investment in the production opportunity and the risk-free instrument
- denote by $\omega_\eta \geq 0$ the proportion of wealth that is invested in the production opportunity, then $(1-\omega_\eta)W_t$ is the amount invested in the risk-free instrument
- the dynamic budget equation is

$$\begin{aligned} dW_t &= (\omega_\eta W_t \mu(x_t,t) + (1-\omega_\eta)W_t r_t - C_t)dt + \omega_\eta W_t \sigma(x_t,t)dz_t \\ &= (\omega_\eta W_t \left(\mu(x_t,t) - r_t\right) + W_t r_t - C_t)dt + \omega_\eta W_t \sigma(x_t,t)dz_t \end{aligned}$$

- the new optimization problem is $0 = \max_{C_t, \omega_\eta} \{\mathbf{\Psi}(C_t, \omega_\eta)\}$ s.t. $C_t \geq 0$, $\omega_\eta \geq 0$ with

$$\begin{aligned}\mathbf{\Psi}(C_t, \omega_\eta) &= U(C_t, t) + \frac{\partial J}{\partial W_t}(\omega_\eta W_t(\mu - r_t) + W_t r_t - C_t) + \frac{\partial J}{\partial x_t} a + \frac{\partial J}{\partial t} \\ &\quad + \frac{1}{2}\frac{\partial^2 J}{\partial W_t^2}\omega_\eta^2 W_t^2 \sigma^2 + \frac{1}{2}\frac{\partial^2 J}{\partial x_t^2} b^2 + \frac{\partial^2 J}{\partial W_t \partial x_t}\omega_\eta W_t \phi\end{aligned}$$

- the FOC for C_t do not change and the FOC for ω_η are

$$\begin{aligned}\frac{\partial \mathbf{\Psi}}{\partial \omega_\eta} &= \frac{\partial J}{\partial W_t} W_t(\mu - r_t) + \frac{\partial^2 J}{\partial W_t^2}\omega_\eta W_t^2 \sigma^2 + \frac{\partial^2 J}{\partial W_t \partial x_t} W_t \phi \leq 0 \\ \omega_\eta \frac{\partial \mathbf{\Psi}}{\partial \omega_\eta} &= \frac{\partial J}{\partial W_t}\omega_\eta W_t(\mu - r_t) + \frac{\partial^2 J}{\partial W_t^2}\omega_\eta^2 W_t^2 \sigma^2 + \frac{\partial^2 J}{\partial W_t \partial x_t}\omega_\eta W_t \phi = 0\end{aligned}$$

- again, the FOC are both necessary and sufficient

- we want to find the optimum conditional on the investor only consuming or investing in the production opportunity but without holding the risk-free asset
- solving the second FOC for r_t and setting $\omega_\eta = 1$ yields

$$r_t = \mu + \frac{\partial^2 J/\partial W_t^2}{\partial J/\partial W_t} W_t \sigma^2 + \frac{\partial^2 J/\partial W_t \partial x_t}{\partial J/\partial W_t} \phi$$

- the equilibrium interest rate r_t depends on
  (i) the instantaneous mean return $\mu(x,t)$ of the optimally invested wealth
  (ii) a term reflecting the uncertainty about the returns of the production opportunity
  (iii) a term reflecting the uncertainty about the state of the technology

- note that by definition

$$\frac{\partial^2 J/\partial W_t^2}{\partial J/\partial W_t} W_t = \gamma - 1$$

and thus

$$r_t = \mu + (\gamma - 1)\sigma^2 + \frac{\partial^2 J/\partial W_t \partial x_t}{\partial J/\partial W_t}\phi$$

- for the special case of logarithmic utility ($\gamma = 0$), the indirect utility function becomes

$$J(W_t, x_t, t) = f(t)\ln(W_t) + g(x_t, t) \quad \Rightarrow \quad \frac{\partial^2 J}{\partial W_t \partial x_t} = 0$$

and we get

$$r_t = \mu(x, t) - \sigma^2(x, t)$$

- under logarithmic utility, the interest rate only depends on the stochastic dynamics of the production opportunity
- r_t is independent of the future production risk arising from the state variable x (i.e. $a(x, t)$ and $b(x, t)$)

V) Equilibrium Dynamics of the Risk-free Rate

-

Model Assumptions

1) **"factor dynamics":**

- the drift and diffusion coefficients or the state variable are $a(x_t, t) = a_0 + a_1 x_t$ and $b(x_t, t) = b_0 \sqrt{x_t}$, i.e.

$$dx_t = (a_0 + a_1 x_t)dt + b_0 \sqrt{x_t} d\zeta_t$$

  where a_0, a_1 and b_0 are constants, $a_0 \geq 0$ and $(dz_t)(d\zeta_t) = \rho dt$
- for $a_0 > 0$ and $a_1 < 0$, x_t is a non-negative mean-reverting random variable
- note that $(dx_t)^2 = b_0^2 x_t dt$

2) **“production dynamics”:**

- the means and variances of the rates of return of the production process are proportional to x_t, i.e. $\mu(x_t, t) = \hat{\mu} x_t$ and $\sigma(x_t, t) = \hat{\sigma}\sqrt{x_t}$

$$\frac{d\eta_t}{\eta_t} = \hat{\mu} x_t dt + \hat{\sigma}\sqrt{x_t} dz_t$$

- given a fixed $x_t = \bar{x}$ this yield a geometric Brownian motion for η_t and thus normally distributed returns

$$\ln\left(\frac{\eta_T}{\eta_0}\right) \sim \mathcal{N}\left(\left[\hat{\mu} - \frac{1}{2}\hat{\sigma}^2\right]\bar{x}T, \hat{\sigma}^2\bar{x}T\right)$$

- technological progress increases both, the mean-return and the variance of the production process

Interest Rate Dynamics

- using the assumption about the factor and production dynamics allows the equilibrium interest rate r_t becomes

$$r_t = \mu(x_t, t) - \sigma^2(x_t, t) = \hat{\mu} x_t - \hat{\sigma}^2 x_t$$

- applying Itô's lemma yields the diffusion process of the risk-free rate

$$\begin{aligned} dr_t &= \left(\hat{\mu} - \hat{\sigma}^2\right) dx_t \\ &= \left(a_0 \left(\hat{\mu} - \hat{\sigma}^2\right) + a_1 r_t\right) dt + b_0 \sqrt{\hat{\mu} - \hat{\sigma}^2} \sqrt{r_t} d\zeta_t \\ &= \kappa \left(\bar{r} - r_t\right) dt + \tilde{\sigma} \sqrt{r_t} d\zeta_t \end{aligned}$$

where $\kappa = -a_1$, $\bar{r} = -\left(a_0 \left(\hat{\mu} - \hat{\sigma}^2\right)\right) a_1^{-1}$ and $\tilde{\sigma} = b_0 \sqrt{\hat{\mu} - \hat{\sigma}^2}$

Stochastic Properties of $\mathbf{r_t}$

(i) r_t follows a square root diffusion that is almost surely non-negative

(ii) the interest rate is elastically pulled towards a long term value $\bar{r} > 0$

(iii) $\kappa > 0$ determines the speed of mean-reversion

(iv) *"Feller condition"*: if $2\kappa\bar{r} \geq \tilde{\sigma}^2$ and $r_t > 0$, then the process does not reach zero almost surely

(v) r_t follows a non-central chi-square distribution

(vi) r_t for $s \leq t$, the conditional mean and variance are

$$
\begin{aligned}
\mathbb{E}\left[r_t | \mathcal{F}_s \right] &= r_s e^{\kappa(t-s)} + \bar{r}\left(1 - e^{-\kappa(t-s)}\right) \\
\mathrm{Var}\left[r_t | \mathcal{F}_s \right] &= \frac{\tilde{\sigma}^2}{\kappa} r_s \left(e^{-\kappa(t-s)} - e^{-2\kappa(t-s)}\right) + \frac{\bar{r}\tilde{\sigma}^2}{2\kappa}\left(1 - e^{-\kappa(t-s)}\right)^2
\end{aligned}
$$

this directly follows from the distributional properties or can be derived using Itô's lemma (see appendix A)

Empirical Properties of Interest Rates that are reflected by $\mathbf{r_t}$

(i) r_t shows mean-reversion

(ii) interest rates can not become negative $\Rightarrow$ r_t is a suitable model for the nominal interest rate

(iii) even if the Feller condition is not fulfilled and $r_t = 0$, then this value is not absorbing

(iv) the absolute variance of the interest rate increases when r_t increases

Sample Paths of the CIR Process

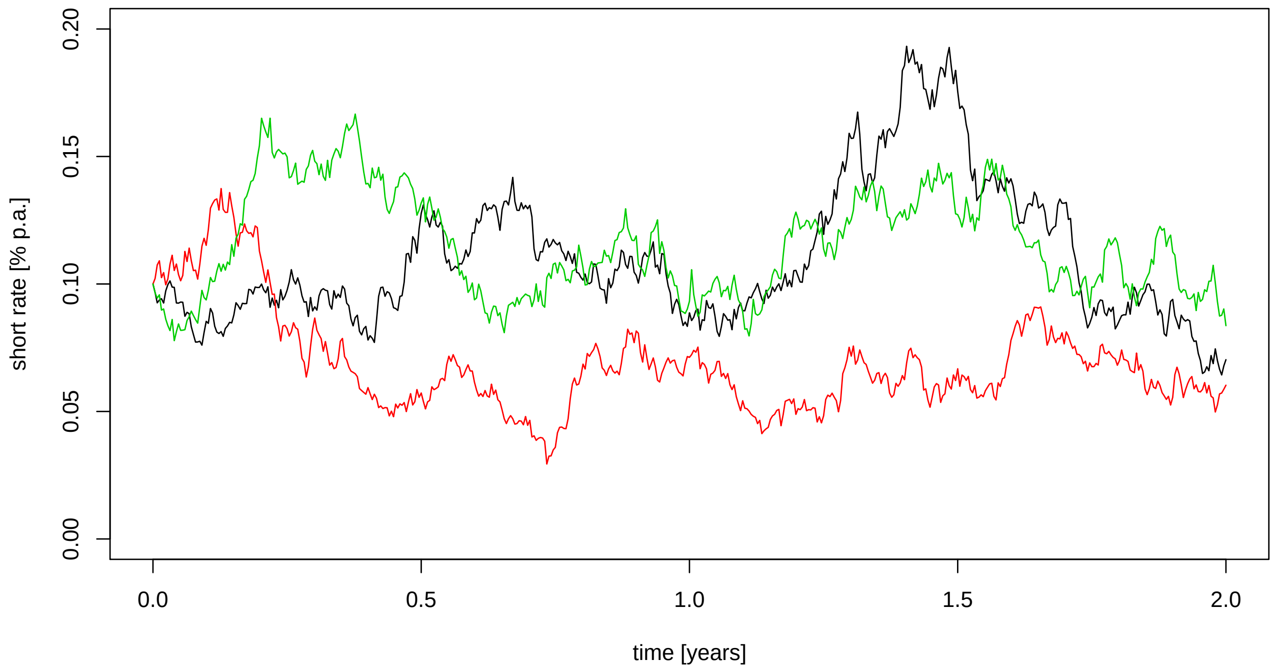

Figure 1: Sample paths for the CIR process with $r_0 = 0.10$, $\kappa = 1.0$, $\bar{r} = 0.10$, $\tilde{\sigma} = 0.20$.

VI) Equilibrium Pricing of Contingent Claims

- just like the risk-free rate, all contingent claims are in zero net supply and thus not held by the representative investor in equilibrium
- denote by $P(W_t, x_t, t)$ the price of a contingent claim
- by Itô's lemma, its differential takes the form

$$
\begin{aligned}
dP_t &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial W_t}dW_t + \frac{\partial P}{\partial x_t}dx_t + \frac{1}{2}\frac{\partial^2 P}{\partial W_t^2}\left(dW_t\right)^2 \\
&\quad + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2}\left(dx_t\right)^2 + \frac{\partial^2 P}{\partial W_t \partial x_t}dW_t dx_t
\end{aligned}
$$

- collecting all drift terms, we set $\alpha(W_t, x_t, t)$ to be the instantaneous return of the derivative security to obtain

$$
dP_t = \alpha(W_t, x_t, t)P_t dt + \frac{\partial P}{\partial W_t}W_t \sigma dz_t + \frac{\partial P}{\partial x_t} b d\zeta_t
$$

- accordingly, we choose

$$\beta_\eta(W_t, x_t, t)P_t = \frac{\partial P}{\partial W_t} W_t \sigma, \quad \beta_x(W_t, x_t, t)P_t = \frac{\partial P}{\partial x_t} b$$

to get the following expression for the instantaneous yield of the contingent claim

$$\frac{dP_t}{P_t} = \alpha(W_t, x_t, t)dt + \beta_\eta(W_t, x_t, t)dz_t + \beta_x(W_t, x_t, t)d\zeta_t$$

- let ω_η be the proportion of wealth that is invested in the production opportunity and ω_P be the proportion invested in the derivative instrument
- $(1 - \omega_\eta - \omega_P)\, W_t$ is the amount invested in the risk-free instrument
- the dynamic budget equation is

$$\begin{aligned} dW_t &= (\omega_\eta W_t(\mu - r_t) + \omega_P W_t(\alpha - r_t) + W_t r_t - C_t)\, dt \\ &\quad + W_t\,(\omega_\eta \sigma + \omega_P \beta_\eta)\, dz_t + W_t \omega_P \beta_x d\zeta_t \end{aligned}$$

- the new optimization problem is $0 = \max_{C_t,\omega_\eta,\omega_P} \{\mathbf{\Psi}(C_t, \omega_\eta, \omega_P)\}$ s.t. $C_t \geq 0$, $\omega_\eta \geq 0$ with

$$
\begin{aligned}
& \mathbf{\Psi}(C_t, \omega_\eta, \omega_P) \\
= \quad & U(C_t, t) \\
& + \frac{\partial J}{\partial W_t}(\omega_\eta W_t(\mu - r_t) + \omega_P W_t(\alpha - r_t) + W_t r_t - C_t) + \frac{\partial J}{\partial x_t} a + \frac{\partial J}{\partial t} \\
& + \frac{1}{2}\frac{\partial^2 J}{\partial W_t^2} W_t^2 \left[(\omega_\eta \sigma + \omega_P \beta_\eta)^2 + (\omega_P \beta_x)^2 + 2(\omega_\eta \sigma + \omega_P \beta_\eta)\, \omega_P \beta_x \rho \right] \\
& + \frac{1}{2}\frac{\partial^2 J}{\partial x_t^2} b^2 + \frac{\partial^2 J}{\partial W_t \partial x_t} W_t \left[(\omega_\eta \sigma + \omega_P \beta_\eta)\, \rho + \omega_P \beta_x \right] b
\end{aligned}
$$

- the new FOC for ω_P is

$$\begin{aligned}\frac{\partial \boldsymbol{\Psi}}{\partial \omega_P} &= \frac{\partial J}{\partial W_t} W_t(\alpha - r_t) \\ &+ \frac{\partial^2 J}{\partial W_t^2} W_t^2 \left[\omega_P \left(\beta_\eta^2 + \beta_x^2 + 2\beta_\eta \beta_x \rho\right) + \omega_\eta \sigma \beta_\eta + \omega_\eta \sigma \beta_x \rho\right] \\ &+ \frac{\partial^2 J}{\partial W_t \partial x_t} W_t \left[\beta_\eta \rho + \beta_x\right] b = 0\end{aligned}$$

- since there is zero net supply for the contingent claim, we set $\omega_\eta = 1$, $\omega_P = 0$ and solve for the equilibrium excess return $\alpha - r_t$ to get

$$\begin{aligned}\alpha - r_t &= \left[-\frac{\partial^2 J/\partial W_t^2}{\partial J/\partial W_t} W_t \sigma - \frac{\partial^2 J/\partial W_t \partial x_t}{\partial J/\partial W_t} b\rho\right] \beta_\eta \\ &+ \left[-\frac{\partial^2 J/\partial W_t^2}{\partial J/\partial W_t} W_t \sigma\rho - \frac{\partial^2 J/\partial W_t \partial x_t}{\partial J/\partial W_t} b\right] \beta_x\end{aligned}$$

Market Prices of Factor Risks

- the equation for the equilibrium excess return $\alpha - r_t$ allows us to define the market prices of factor risks as

$$\theta_\eta = -\frac{\partial^2 J/\partial W_t^2}{\partial J/\partial W_t} W_t \sigma - \frac{\partial^2 J/\partial W_t \partial x_t}{\partial J/\partial W_t} b\rho$$

$$\theta_x = -\frac{\partial^2 J/\partial W_t^2}{\partial J/\partial W_t} W_t \sigma \rho - \frac{\partial^2 J/\partial W_t \partial x_t}{\partial J/\partial W_t} b$$

- once again, note that under iso-elastic utility

$$-\frac{\partial^2 J/\partial W_t^2}{\partial J/\partial W_t} W_t = 1 - \gamma, \quad \frac{\partial^2 J/\partial W_t \partial x_t}{\partial J/\partial W_t} = 0$$

- $\alpha - r_t$ is the excess return that makes the representative investor indifferent between buying and selling the contingent claim

- θ_η and θ_x relate the excess return to the amount of risk taken in β_η and β_x
- just like the equilibrium risk-free rate, the market prices of factors risks are independent of the aggregate wealth in the economy
- equilibrium prices of derivatives whose payoffs are independent of wealth are thus independent of W_t themselves
- as shown earlier, for the special case of logarithmic utility ($\gamma = 0$), the RRA is constant at one and the market prices of risk simplify to

$$\theta_\eta = \sigma(x, t), \quad \theta_x = \sigma(x, t)\rho$$

- the market prices of risk do not only depend on the investor's utility function but also on the dynamics of the production opportunity and the state variable as well as their correlation

Fundamental Valuation Equation

- by Itô's lemma, the differential of a contingent claim $P(x,t)$ with wealth-independent payoff takes the form

$$\begin{aligned} dP_t &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial x_t}dx_t + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2}\left(dx_t\right)^2 \\ &= \left(\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x_t}a(x,t) + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2}b^2(x,t)\right)dt + \frac{\partial P}{\partial x_t}b(x,t)d\zeta_t \end{aligned}$$

- using the previous, the instantaneous drift $\alpha(x,t)$ and the instantaneous diffusion $\beta_x(x,t)$ are

$$\begin{aligned} \alpha(x,t) &= \frac{1}{P_t}\left(\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x_t}a(x,t) + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2}b^2(x,t)\right) \\ \beta_x(x,t) &= \frac{1}{P_t}\left(\frac{\partial P}{\partial x_t}b(x,t)\right) \end{aligned}$$

- the contingent claim's dynamics can then be expressed as

$$dP_t = \alpha(x,t)P_t dt + \beta_x(x,t)P_t d\zeta_t$$

- plugging these expressions for $\alpha(x,t)$ and $\beta(x,t)$ into the equation for the equilibrium excess return yields the *"fundamental valuation equation"*

$$\begin{aligned}
& \alpha(x,t) - r_t = \theta_x \beta_x(x,t) \\
\Rightarrow \quad & \frac{1}{P_t}\left(\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x_t}a(x,t) + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2}b^2(x,t)\right) - r_t = \frac{\sigma(x,t)}{P_t}\left(\frac{\partial P}{\partial x_t}b(x,t)\right) \\
\Rightarrow \quad & \frac{\partial P}{\partial t} + \frac{\partial P}{\partial x_t}\left[a(x,t) - \sigma(x,t)b(x,t)\rho\right] + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2}b^2(x,t) = rP_t
\end{aligned}$$

Valuation of Zero-Bonds

- let $P(x_t, t, T)$ be the time t price of a default-free zero bond that pays one unit of the consumption good in $T \geq t$
- note that the payoff of the zero bond is independent of wealth $\Rightarrow$ its equilibrium dynamics can be described by the fundamental valuation equation when utility is logarithmic

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x_t}\left[a(x_t, t) - \sigma(x_t, t) b(x_t, t)\rho\right] + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2} b^2(x_t, t) = r_t P_t$$

- we replace the general drift and diffusion terms by the concrete choice that has been made to derive the CIR process ($a(x_t, t) = a_0 + a_1 x_t$, $b(x_t, t) = b_0 \sqrt{x_t}$ and $\sigma(x_t, t) = \hat{\sigma}\sqrt{x_t}$) to get

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x_t}(a_0 + a_1 x_t - \hat{\sigma} b_0 \rho x_t) + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2} b_0^2 x_t = r_t P_t$$

- since the equilibrium interest rate is $r_t = \left(\hat{\mu} - \hat{\sigma}^2\right) x_t$, we can apply a change of variable from $P(x_t, t, T)$ to $P(r_t, t, T)$ where

$$\frac{\partial P}{\partial x_t} = \frac{\partial P}{\partial r_t}\left(\hat{\mu} - \hat{\sigma}^2\right), \quad \frac{\partial^2 P}{\partial x_t^2} = \frac{\partial^2 P}{\partial r_t^2}\left(\hat{\mu} - \hat{\sigma}^2\right)^2$$

and obtain

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r_t}\left(a_0\left(\hat{\mu} - \hat{\sigma}^2\right) + a_1 r_t - \hat{\sigma} b_0 \rho r_t\right) + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2} b_0^2 \left(\hat{\mu} - \hat{\sigma}^2\right) r_t = r_t P_t$$

- finally, using the notation of the interest rate dynamics and setting $\psi = \hat{\sigma} b_0 \rho$ yields

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r_t}\left(\kappa\left(\bar{r} - r_t\right) - \psi r_t\right) + \frac{1}{2}\frac{\partial^2 P}{\partial x_t^2}\tilde{\sigma} r_t = r_t P_t$$

- ψr_t is the covariance of interest rate changes with the proportional change in optimally invested wealth (the interest rate's "beta")

- the boundary condition for the zero bond is $P(r_t, T, T) = 1$
- the CIR process falls into the general class of affine term structure models where the drift and the squared diffusion term are affine functions of r_t
- there exists a closed form solution for zero bond prices under affine term structure models and for the case of the CIR model it is

$$P(r, t, T) = A(\tau) e^{-B(\tau) r_t}, \quad \tau = T - t$$

where

$$\begin{aligned} A(\tau) &= \left[\frac{2\theta e^{(\theta+\kappa+\psi)\frac{\tau}{2}}}{(\theta+\kappa+\psi)\left(e^{\theta\tau}-1\right)+2\theta} \right]^{2\kappa\bar{r}/\tilde{\sigma}^2} \\ B(\tau) &= \frac{2\left(e^{\theta\tau}-1\right)}{(\theta+\kappa+\psi)\left(e^{\theta\tau}-1\right)+2\theta} \\ \theta &= \sqrt{(\kappa+\psi)^2 + 2\tilde{\sigma}^2} \end{aligned}$$

Comparative Statics

parameter	sensitivity of $P(r,t,T)$
$r_t \uparrow$	$\downarrow$ (convex)
$\tau \uparrow$	$\downarrow$
$\theta \uparrow$	$\downarrow$ (convex)
$\kappa \uparrow, r_t > \theta$	$\uparrow$ (concave)
$\kappa \uparrow, r_t < \theta$	$\downarrow$ (convex)
$\psi \uparrow$	$\uparrow$ (concave)
$\sigma^2 \uparrow$	$\uparrow$ (concave)

Term Structure of Interest Rates for different $\mathbf{r_t}$

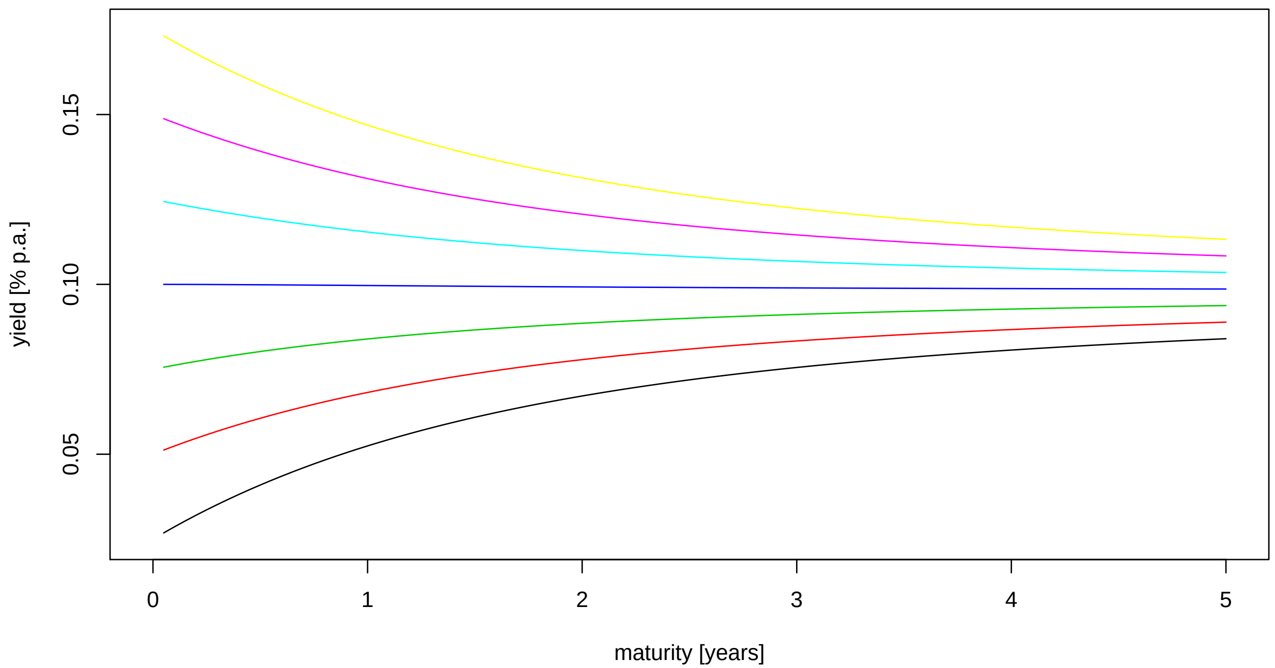

Figure 2: Term structures of interest rates for $r_0 \in \{0.025, 0.050, \ldots, 0.175\}$, $\kappa = 1.0$, $\bar{r} = 0.10$, $\tilde{\sigma} = 0.20$.

Term Structure of Interest Rates for different κ

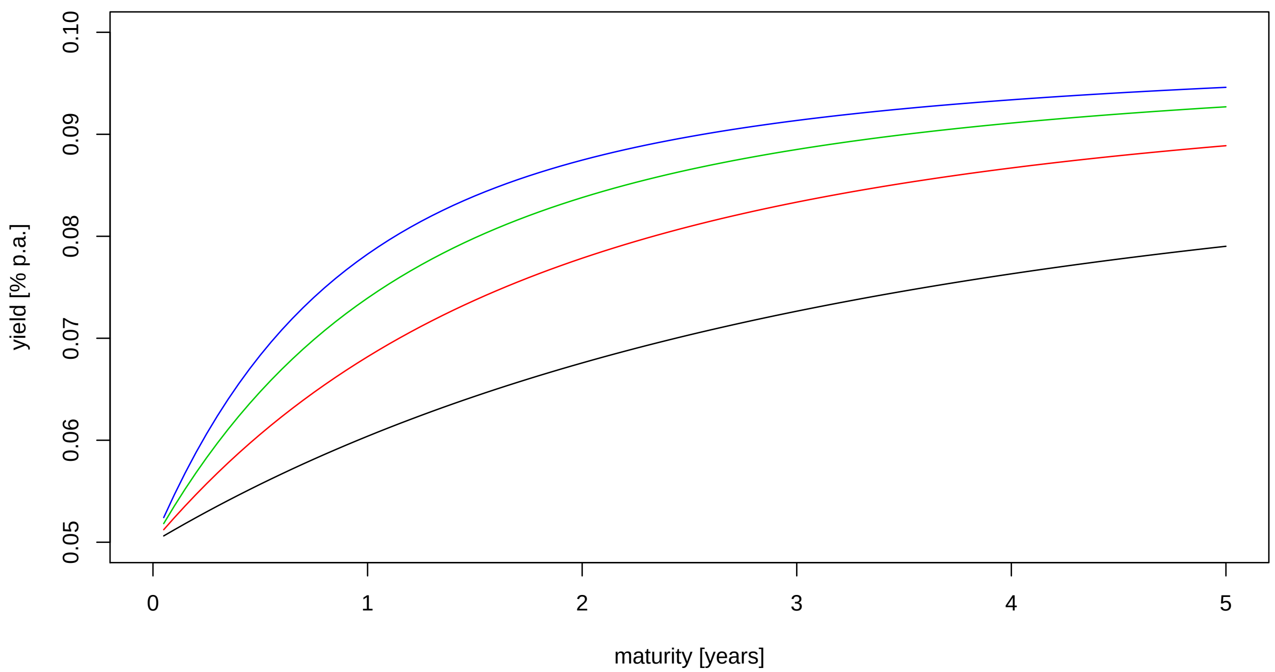

Figure 3: Term structures of interest rates for $r_0 = 0.05$, $\kappa \in \{0.5, 1.0, 1.5, 2.0\}$, $\bar{r} = 0.10$, $\tilde{\sigma} = 0.20$.

VII) In Contrast: the Arbitrage Approach

- the dynamics of the state variable x_t and the risk-free rate are given exogenously
- derivative instruments are priced relative to given market prices
- the market price of risk (MPR) relates the excess return to the amount of risk taken and is determined exogenously

$$\alpha - r_t = \theta_x(x_t, t)\beta_x$$

- if no exogenous prices are available that can be used to determine the MPR, one has to impose assumptions about its functional form
- closing the model by choosing a certain function form of θ_x might result in internal inconsistencies and a model that is not free of arbitrage

- the equilibrium approach determines the MPR endogenously within the specific model of the economy
- not all functional forms of θ_x can be obtained within an equilibrium model but all endogenously determined MPR yield a pricing model that is free of arbitrage

Thank you!

-

Questions?

Appendix A

-

Derivation of the first two Moments of the CIR Process

- we note that there is negative geometric drift proportional to κ and thus set $X_t = f(t, x) = e^{\kappa t} x$ to obtain

$$
\begin{aligned}
df(t,x) &= f_t(t,r_t)dt + f_x(t,r_t)dr_t + \frac{1}{2} f_{xx}(t,r_t)\left(dr_t\right)^2 \\
&= \kappa e^{\kappa t} r_t dt + e^{\kappa t} \kappa \left(\bar{r} - r_t\right) dt + e^{\kappa t} \tilde{\sigma} \sqrt{r_t} d\zeta_t \\
&= e^{\kappa t} \kappa \bar{r} dt + e^{\kappa t} \tilde{\sigma} \sqrt{r_t} d\zeta_t
\end{aligned}
$$

- integrating both sides yields

$$\begin{aligned} e^{\kappa t} r_t &= e^{\kappa s} r_s + \kappa \bar{r} \int_s^t e^{\kappa \tau} d\tau + \sigma \int_s^t \sqrt{r_\tau} d\zeta_t \\ &= e^{\kappa s} r_s + \bar{r} \left(e^{\kappa t} - e^{\kappa s} \right) + \tilde{\sigma} \int_s^t \sqrt{r_\tau} d\zeta_t \end{aligned}$$

- since an Itô integral has zero expected value, we get

$$\mathbb{E}\left[r_t \middle| \mathcal{F}_s \right] = r_s e^{-\kappa(t-s)} + \bar{r} \left(1 - e^{-\kappa(t-s)} \right)$$

- we get the following limiting relationships for the mean-reversion factor κ

$$\lim_{\kappa \to \infty} \mathbb{E}\left[r_t \middle| \mathcal{F}_s \right] = \bar{r}, \quad \lim_{\kappa \to 0} \mathbb{E}\left[r_t \middle| \mathcal{F}_s \right] = r_s$$

- note that dX_t as computed above can be expressed as

$$dX_t = e^{\kappa t} \kappa \bar{r} dt + e^{\frac{\kappa t}{2}} \tilde{\sigma} \sqrt{X_t} d\zeta_t$$

- applying the Itô formula to compute $d\left(X_t\right)^2$ yields

$$\begin{aligned} d\left(X_t\right)^2 &= 2X_t dX_t + \left(dX_t\right)^2 \\ &= 2e^{\kappa t}\kappa\bar{r}X_t dt + 2e^{\frac{\kappa t}{2}}\tilde{\sigma}X_t\sqrt{X_t}d\zeta_t + e^{\kappa t}\tilde{\sigma}^2 X_t dt \end{aligned}$$

- integrating both sides gives us

$$X_t^2 = X_s^2 + \left(2\kappa\bar{r} + \tilde{\sigma}^2\right)\int_s^t e^{\kappa\tau}X_\tau d\tau + 2\tilde{\sigma}\int_s^t e^{\frac{\kappa\tau}{2}}X_\tau\sqrt{X_\tau}d\zeta_\tau$$

- when taking the expected value, the Itô integrals drops out again

$$\begin{aligned} \mathbb{E}\left[\left. X_t^2 \right| \mathcal{F}_s\right] &= X_s^2 + \left(2\kappa\bar{r} + \tilde{\sigma}^2\right)\int_s^t e^{\kappa\tau}\mathbb{E}\left[\left. X_\tau \right| \mathcal{F}_s\right] d\tau \\ &= X_s^2 + \left(2\kappa\bar{r} + \tilde{\sigma}^2\right)\int_s^t e^{\kappa\tau}\left[r_s e^{\kappa s} + \bar{r}\left(e^{\kappa\tau} - e^{\kappa s}\right)\right] d\tau \end{aligned}$$

- solving the integral gives a closed form solution for the expected value of X_t^2

$$\mathbb{E}\left[X_t^2\middle|\mathcal{F}_s\right]=X_s^2+\frac{2\kappa\bar{r}+\tilde{\sigma}^2}{\kappa}e^{\kappa s}\left(r_s+\bar{r}_t\right)\left(e^{\kappa t}-e^{\kappa s}\right)+\frac{2\kappa\bar{r}+\tilde{\sigma}^2}{2\kappa}\bar{r}\left(e^{2\kappa t}-e^{2\kappa s}\right)$$

- back substitution of $\mathbb{E}\left[r_t^2\middle|\mathcal{F}_s\right]=e^{-2\kappa t}\mathbb{E}\left[X_t^2\middle|\mathcal{F}_s\right]$ yields

$$\begin{aligned}\mathbb{E}\left[r_t^2\middle|\mathcal{F}_s\right] &= e^{-2\kappa(t-s)}r_s+\frac{2\kappa\bar{r}+\tilde{\sigma}^2}{\kappa}\left(r_s+\bar{r}_t\right)\left(e^{-\kappa(t-s)}-e^{-2\kappa(t-s)}\right)\\ &\quad+\frac{2\kappa\bar{r}+\tilde{\sigma}^2}{2\kappa}\bar{r}\left(1-e^{-2\kappa(t-s)}\right)\end{aligned}$$

- the variance of r_t can then be computes as Var $\left[r_t\middle|\mathcal{F}_s\right]=\mathbb{E}\left[r_t^2\middle|\mathcal{F}_s\right]-\left(\mathbb{E}\left[r_t\middle|\mathcal{F}_s\right]\right)^2$

$$\mathrm{Var}\left[r_t\middle|\mathcal{F}_s\right]=\frac{\tilde{\sigma}^2}{\kappa}r_s\left(e^{-\kappa(t-s)}-e^{-2\kappa(t-s)}\right)+\frac{\bar{r}\tilde{\sigma}^2}{2\kappa}\left(1-e^{-\kappa(t-s)}\right)^2$$