A Course in Financial Calculus
-
some Solutions

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Exercise 4.12

i) Let $f(t,x) = \sqrt{x}$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x}}, \quad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{4x\sqrt{x}}$$

and

$$(dr_t)^2 = \sigma^2 r_t dt.$$

Applying Itô's lemma for $\alpha = 0$, the differential of $\sqrt{r_t}$ becomes

$$\begin{aligned} d\sqrt{r_t} &= -\frac{1}{2}\beta\sqrt{r_t}dt + \frac{1}{2}\sigma dW_t - \frac{1}{8}\frac{\sigma^2}{\sqrt{r_t}}dt \\ &= -\frac{1}{2}\left(\beta\sqrt{r_t} + \frac{1}{4}\frac{\sigma^2}{\sqrt{r_t}}\right)dt + \frac{1}{2}\sigma dW_t. \end{aligned}$$

ii) Let $\tau = T - t$. Applying a change of variable from $u(t)$ to $u(\tau)$ yields

$$\frac{du(t)}{dt} = \frac{du(\tau)}{d\tau}\frac{d\tau}{dt} = -\frac{du(\tau)}{d\tau}$$

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and we obtain

$$\frac{du(\tau)}{d\tau} = \beta u(\tau) + \frac{\sigma^2}{2}u^2(\tau).$$

Now let $f(t,x,y) = \exp\{-x(t)y\}$. We first note that $(du(\tau))^2 = 0$ and $du(\tau)dr_t = 0$ since $u(\tau)$ is a non-random function of time. The relevant partial derivatives are

$$\frac{\partial f}{\partial x} = -yf(t,x,y), \quad \frac{\partial f}{\partial y} = -x(t)f(t,x,y), \quad \frac{\partial^2 f}{\partial y^2} = x^2(t)f(t,x,y).$$

We set $Y_t = \exp\{-u(T-t)r_t\}$, use that $\alpha = 0$ and apply Iô's lemma to obtain

$$\begin{aligned}
dY_t &= -r_tY_tdu(T-t) - u(T-t)Y_tdr_t + \frac{1}{2}u^2(T-t)Y_t\,(dr_t)^2 \\
&= -r_tY_t\beta u(T-t)dt - r_tY_t\frac{\sigma^2}{2}u^2(T-t)dt + u(T-t)Y_t\beta r_tdt \\
&\quad -u(T-t)Y_t\sigma\sqrt{r_t}dW_t + \frac{1}{2}u^2(T-t)Y_t\sigma^2r_tdt \\
&= -u(T-t)Y_t\sigma\sqrt{r_t}dW_t \\
Y_t &= Y_0 - \int_0^t u(T-s)Y_s\sigma\sqrt{r_s}dW_s.
\end{aligned}$$

Since Itô integrals are martingales with zero initial value, it follows that the differential of the expected value is

$$\begin{aligned}
d\mathbb{E}\left[Y_T\right] &= 0 \\
\mathbb{E}\left[Y_T\right] &= \mathbb{E}\left[\exp\{-u(0)r_T\}\right] = Y_0 = \exp\{-u(T)r_0\}. \qquad (1)
\end{aligned}$$

Since $u(0) = \theta$ is given, we only need to solve the ordinary differential equation that defines $u(t)$ to obtain the moment generating function of r_T and account for the negative sign inside the exponential. Since the ODE is a Bernoulli first-order equation, we start by defining

$$v(t) = \frac{1}{u(t)} \quad \Rightarrow \quad u(t) = \frac{1}{v(t)}, \quad du(t) = -\frac{1}{v^2(t)}dv(t).$$

Substituting into the ODE yields

$$\begin{aligned}
-\frac{1}{v^2(t)}\frac{dv(t)}{dt} &= -\frac{\beta}{v(t)} - \frac{\sigma^2}{2v^2(t)} \\
\frac{dv(t)}{dt} &= \beta v(t) + \frac{\sigma^2}{2}. \qquad (2)
\end{aligned}$$

Equation (2) is now a linear ODE and we multiply both sides by the integration factor $e^{-\beta t}$ to obtain

$$\begin{aligned}
e^{-\beta t}\frac{dv(t)}{dt} - \beta e^{-\beta t} v(t) &= e^{-\beta t}\frac{\sigma^2}{2} \\
\frac{d\left(e^{-\beta t} v(t)\right)}{dt} &= e^{-\beta t}\frac{\sigma^2}{2}.
\end{aligned}$$

We can now integrate both sides to obtain the solution for $v(t)$

$$\begin{aligned}
\int d\left(e^{-\beta t} v(t)\right) &= \int e^{-\beta t}\frac{\sigma^2}{2} dt \\
e^{-\beta t} v(t) &= -\frac{\sigma^2}{2\beta} e^{-\beta t} + C.
\end{aligned}$$

We substitute back to obtain the solution for $u(t)$

$$u(t) = \frac{1}{-\frac{\sigma^2}{2\beta} + e^{\beta t} C}. \qquad (3)$$

Using $u(0) = \theta$ allows us to solve for the integration constant C.

$$\theta = \frac{1}{-\frac{\sigma^2}{2\beta} + C} \quad \Leftrightarrow \quad C = \frac{1 + \frac{\sigma^2}{2\beta}\theta}{\theta}$$

Substituting back into Equation (3) and simplifying yields

$$\begin{aligned}
u(t) &= \frac{1}{-\frac{\sigma^2}{2\beta} + e^{\beta t}\frac{1+\frac{\sigma^2}{2\beta}\theta}{\theta}} \\
&= \frac{e^{-\beta t}\theta}{1 + \frac{\sigma^2}{2\beta}\theta - e^{-\beta t}\frac{\sigma^2}{2\beta}\theta}.
\end{aligned}$$

Equation (1) can now be written as

$$\mathbb{E}\left[\exp\left\{-\theta r_T\right\}\right] = \exp\left\{-\frac{e^{-\beta T}\theta}{1+\frac{\sigma^2}{2\beta}\theta - e^{-\beta T}\frac{\sigma^2}{2\beta}\theta} r_0\right\}. \tag{4}$$

The first two derivatives of this expression w.r.t. θ are

$$\begin{aligned}
\frac{\partial}{\partial\theta}\mathbb{E}\left[\exp\left\{-\theta r_T\right\}\right] &= -\frac{e^{-\beta T}\left(1+\frac{\sigma^2}{2\beta}\theta - e^{-\beta T}\frac{\sigma^2}{2\beta}\theta\right) - e^{-\beta T}\theta\left(\frac{\sigma^2}{2\beta} - e^{-\beta T}\frac{\sigma^2}{2\beta}\right)}{\left(1+\frac{\sigma^2}{2\beta}\theta - e^{-\beta T}\frac{\sigma^2}{2\beta}\theta\right)^2} r_0 \exp\left\{\dots\right\} \\
&= -\frac{e^{-\beta T}}{\left(1+\frac{\sigma^2}{2\beta}\theta - e^{-\beta T}\frac{\sigma^2}{2\beta}\theta\right)^2} r_0 \exp\left\{\dots\right\} \\
\frac{\partial^2}{\partial\theta^2}\mathbb{E}\left[\exp\left\{-\theta r_T\right\}\right] &= \frac{2e^{-\beta T}\left(\frac{\sigma^2}{2\beta} - e^{-\beta T}\frac{\sigma^2}{2\beta}\right)}{\left(1+\frac{\sigma^2}{2\beta}\theta - e^{-\beta T}\frac{\sigma^2}{2\beta}\theta\right)^3} r_0 \exp\left\{\dots\right\} \\
&\quad + \frac{e^{-2\beta T}}{\left(1+\frac{\sigma^2}{2\beta}\theta - e^{-\beta T}\frac{\sigma^2}{2\beta}\theta\right)^4} r_0^2 \exp\left\{\dots\right\}.
\end{aligned}$$

We note once again, that Equations (1) and (4) are not moment generating functions due to the minus sign in the exponential. We thus define $\hat{\theta} = -\theta$ such that $\mathbb{E}\left[\exp\left\{\hat{\theta} r_T\right\}\right]$ is the moment generating function. We furthermore note that

$$\begin{aligned}
\frac{\partial}{\partial\hat{\theta}}\mathbb{E}\left[\exp\left\{\hat{\theta} r_T\right\}\right] &= \frac{\partial}{\partial\theta}\mathbb{E}\left[\exp\left\{\theta r_T\right\}\right]\frac{\partial\theta}{\partial\hat{\theta}} = -\frac{\partial}{\partial\theta}\mathbb{E}\left[\exp\left\{\theta r_T\right\}\right] \\
\frac{\partial^2}{\partial\hat{\theta}^2}\mathbb{E}\left[\exp\left\{\hat{\theta} r_T\right\}\right] &= -\frac{\partial^2}{\partial\theta^2}\mathbb{E}\left[\exp\left\{\theta r_T\right\}\right]\frac{\partial\theta}{\partial\hat{\theta}} = \frac{\partial^2}{\partial\theta^2}\mathbb{E}\left[\exp\left\{\theta r_T\right\}\right].
\end{aligned}$$

Evaluating these two derivatives at $\hat{\theta} = 0$ yields the first two moments

$$\begin{aligned}
\mathbb{E}\left[r_T\right] &= \left.\frac{\partial}{\partial\hat{\theta}}\mathbb{E}\left[\exp\left\{\hat{\theta} r_T\right\}\right]\right|_{\hat{\theta}=0} = e^{-\beta T} r_0 \\
\mathbb{E}\left[r_T^2\right] &= \left.\frac{\partial^2}{\partial\hat{\theta}^2}\mathbb{E}\left[\exp\left\{\hat{\theta} r_T\right\}\right]\right|_{\hat{\theta}=0} = e^{-\beta T}\frac{\sigma^2}{\beta}\left(1-e^{-\beta T}\right) r_0 + e^{-2\beta T} r_0^2.
\end{aligned}$$

The variance of r_T is then given by

$$\begin{aligned}
\mathrm{Var}\left[r_T\right] &= \mathbb{E}\left[r_T^2\right] - \left(\mathbb{E}\left[r_T\right]\right)^2 \\
&= e^{-\beta T}\frac{\sigma^2}{\beta}\left(1-e^{-\beta T}\right) r_0.
\end{aligned}$$

In order to compute the probability $\mathbb{P}[r_T = 0]$, we first need to determine the type of distribution that r_T follows. We start by making the substitution $\bar{\theta} = -\frac{\sigma^2}{2\beta}\theta$ in Equation (4) to get

$$\mathbb{E}\left[\exp\left\{\frac{2\beta}{\sigma^2}\bar{\theta}r_T\right\}\right] = \exp\left\{\frac{e^{-\beta T}\frac{2\beta}{\sigma^2}\bar{\theta}}{1-\bar{\theta}+e^{-\beta T}\bar{\theta}}r_0\right\}.$$

Next, we make the substitution $\tilde{\theta} = \frac{1}{2}\bar{\theta}\left(1-e^{-\beta T}\right)$, i.e. $\bar{\theta} = 2\tilde{\theta}\left(1-e^{-\beta T}\right)^{-1}$ and obtain

$$\mathbb{E}\left[\exp\left\{\frac{4\beta}{\sigma^2\left(1-e^{-\beta T}\right)}\tilde{\theta}r_T\right\}\right] = \exp\left\{\frac{\frac{4\beta e^{-\beta T}}{\sigma^2\left(1-e^{-\beta T}\right)}\tilde{\theta}}{1-2\tilde{\theta}}r_0\right\}.$$

Finally, setting $\lambda = 4\beta e^{-\beta T}\left(\sigma^2\left(1-e^{-\beta T}\right)\right)^{-1}r_0$, we get

$$\mathbb{E}\left[\exp\left\{\frac{\lambda}{e^{-\beta T}r_0}\tilde{\theta}r_T\right\}\right] = \exp\left\{\frac{\lambda\tilde{\theta}}{1-2\tilde{\theta}}\right\}.$$

We require $r_0 \neq 0$ and see that the moment generating function of $\lambda r_T\left(e^{-\beta T}r_0\right)^{-1}$ is the same as the moment generating function of a non-central chi-square distribution with zero degrees of freedom and non-centrality parameter $\lambda \geq 0$. The uniqueness of the moment generating function implies that $\lambda r_T\left(e^{-\beta T}r_0\right)^{-1}$ has a non-central chi-square distribution. As shown by Siegel (1979)[1], the discrete probability mass at zero is given by

$$\mathbb{P}\left[\frac{\lambda r_T}{e^{-\beta T}r_0} = 0\right] = e^{-\frac{1}{2}\lambda} = \exp\left\{-\frac{2\beta e^{-\beta T}r_0}{\sigma^2\left(1-e^{-\beta T}\right)}\right\}.$$

Given that $\beta > 0$, we have $\lambda > 0$ and the scaling has no impact on the probability mass at zero. In this case, we can conclude that

$$\mathbb{P}[r_T = 0] = \exp\left\{-\frac{2\beta e^{-\beta T}r_0}{\sigma^2\left(1-e^{-\beta T}\right)}\right\}.$$

[1]Siegel, Andrew F.: The Noncentral Chi-Squared Distribution with Zero Degrees of Freedom and Testing for Uniformity; Biometrika, Vol. 66, No. 2 (August 1979), pp. 381-386

Exercise 5.1

The portfolio value at time t is given by $V_t = m_t B_t + n_t S_t$. By definition, a portfolio is self-financing if

$$dV_t = n_t dS_t + m_t dB_t = n_t dS_t + r m_t B_t dt. \tag{5}$$

i) We have $n_t = 1$ and search for a predictable process m_t such that the portfolio is self-financing. Applying Itô's product rule, we can compute the differential of the portfolio value with $n_t = 1$ for general m_t as

$$\begin{aligned} dV_t &= dS_t + B_t dm_t + m_t dB_t + (dm_t)(dB_t) \\ &= dS_t + B_t dm_t + r m_t B_t dt. \end{aligned} \tag{6}$$

Here, we have used that the cross variation of m_t and B_t is always zero since B_t is a non-random process of time. In order to have equality between Equations (6) and (5), we thus require $dm_t = 0$. It follows that m_t has to be a constant, i.e. $m_t = m_0 \in \mathbb{R}$.

ii) We follow the same steps as in part (i) and start by computing the differential of the portfolio value with $n_t = \int_0^t S_u du$ for general m_t

$$\begin{aligned} dV_t &= S_t dn_t + n_t dS_t + (dn_t)(dS_t) + B_t dm_t + m_t dB_t + (dm_t)(dB_t) \\ &= S_t^2 dt + n_t dS_t + B_t dm_t + r m_t B_t dt. \end{aligned} \tag{7}$$

The self-financing condition is the same as in part (i). In order to have equality between Equation (7) and (5), we require

$$S_t^2 dt + B_t dm_t = 0 \qquad \Leftrightarrow \qquad m_t = m_0 - \int_0^t S_u^2 e^{-ru} du.$$

Exercise 6.17

(i) The American digital call has fixed payoff of 1\$ if the barrier is being triggered before maturity and zero otherwise. It follows that its price has to satisfy $V_t \leq 1$.

Furthermore, if $S_t \geq K$, the immediate exercise of the call would pay the owner the current intrinsic value of 1\$ and thus its price has to satisfy $V_t \geq 1$ in this situation. Combining these two arguments, we see that the price of the American call has to be equal to the intrinsic value of 1\$ if $S_t \geq K$. Since it is optimal to exercise American derivative securities, when their price is equal to the intrinsic value, it follows that it is optimal to exercise the American digital call at the first time that the level K is reached.

(ii) We first need to derive the first passage time density of a drifted Brownian motion to an upper level $m > 0$. From Theorem 3.7.1 in Steven Shreve's book "Stochastic Calculus for Finance II", we know that the first passage time of a non-drifted Brownian motion to this level has the distribution function and density

$$\mathbb{P}\left\{\tau_m \leq t\right\} = \frac{2}{2\pi}\int_{\frac{m}{\sqrt{t}}}^{\infty} e^{-\frac{y^2}{2}}dy, \quad f_{\tau_m}(t) = \frac{m}{t\sqrt{2\pi t}}e^{-\frac{m^2}{2t}}$$

Let $\tilde{W}(t)$ be a non-drifted Brownian motion under the risk-neutral measure $\tilde{\mathbb{P}}$. We now define a new Brownian motion $\hat{W}(t)$ by

$$\hat{W}(t) = \tilde{W}(t) + \theta t, \qquad \theta \in \mathbb{R}$$

Note that $\hat{W}(t)$ has a non-zero drift under the risk-neutral measure. Define the Radon-Nikodým derivative process $Z(t)$ by

$$Z(t) = \exp\left\{-\theta\tilde{W}(t) - \frac{1}{2}\theta^2 t\right\}$$

By Girsanov's Theorem, $\hat{W}(t)$ is a Brownian motion under the probability measure $\hat{\mathbb{P}}$ defined by

$$\hat{\mathbb{P}}(A) = \int_A Z(t)d\tilde{\mathbb{P}} \qquad \forall A \in \Omega$$

It thus follows that

$$
\begin{aligned}
\mathbb{P}\{\tau_m \le t\} &= \tilde{\mathbb{E}}\left[\mathbb{I}_{\{\tau_m \le t\}}\right] \\
&= \hat{\mathbb{E}}\left[\frac{1}{Z(T)}\mathbb{I}_{\{\tau_m \le t\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left\{\theta\hat{W}(T) - \frac{1}{2}\theta^2 T\right\}\mathbb{I}_{\{\tau_m \le t\}}\right] \\
&= \hat{\mathbb{E}}\left[\mathbb{E}\left[\exp\left\{\theta\hat{W}(T) - \frac{1}{2}\theta^2 T\right\}\middle|\mathcal{F}(\tau_m \wedge t)\right]\mathbb{I}_{\{\tau_m \le t\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left\{\theta\hat{W}(\tau_m \wedge t) - \frac{1}{2}\theta^2(\tau_m \wedge t)\right\}\mathbb{I}_{\{\tau_m \le t\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left\{\theta m - \frac{1}{2}\theta^2 \tau_m\right\}\mathbb{I}_{\{\tau_m \le t\}}\right] \\
&= \int_0^t \frac{m}{s\sqrt{2\pi s}}\exp\left\{-\frac{m^2 - 2m\theta s - \theta^2 s^2}{2s}\right\}ds \\
&= \int_0^t \frac{m}{s\sqrt{2\pi s}}\exp\left\{-\frac{(m-\theta s)^2}{2s}\right\}ds
\end{aligned}
$$

Taking the first derivative w.r.t. t finally yields the first passage time density of $\hat{W}(t)$ to the level m under $\tilde{\mathbb{P}}$

$$
\hat{f}_{\tau_m}(t) = \frac{m}{t\sqrt{2\pi t}}\exp\left\{-\frac{(m-\theta t)^2}{2t}\right\}
$$

(iii) In order to determine the level m, we first set

$$
\theta = \frac{r - \frac{1}{2}\sigma^2}{\sigma}.
$$

The solution to the geometric Brownian motion SDE is then given by

$$
S(t) = S(0)\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{W}(t)\right\} = S(0)\exp\left\{\sigma\hat{W}(t)\right\}
$$

The spot price at time t triggers the barrier if

$$
S(t) \ge B \quad \Leftrightarrow \quad \hat{W}(t) \ge \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma} = m
$$

Under the risk-neutral measure, the current price of the American digital call is the expected discount factor of the first passage time to the barrier level m conditional on the barrier being triggered before the maturity. We get

$$\begin{aligned} V(0) &= \tilde{\mathbb{E}}\left[e^{-r\tau_m}\mathbb{I}_{\{\tau_m \le T\}}\right] \\ &= \int_0^T e^{-rt}\frac{m}{t\sqrt{2\pi t}}\exp\left\{-\frac{(m-\theta t)^2}{2t}\right\}dt \end{aligned}$$

Now let $\alpha = \sigma\theta = r - \frac{1}{2}\sigma^2$. We substitute for m to get

$$\begin{aligned} \ldots &= \int_0^T e^{-rt}\frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right)-\alpha t\right)^2}{2\sigma^2 t}\right\}dt \\ &= \int_0^T e^{-rt}\frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\ln^2\left(\frac{B}{S(0)}\right)-2\ln\left(\frac{B}{S(0)}\right)\alpha t+\alpha^2 t^2}{2\sigma^2 t}\right\}dt \\ &= \left(\frac{B}{S_0}\right)^{\frac{\alpha}{\sigma^2}}\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\ln^2\left(\frac{B}{S(0)}\right)+\left(\alpha^2+2\sigma^2 r\right)t^2}{2\sigma^2 t}\right\}dt \end{aligned}$$

Now let $\beta = \sqrt{\alpha^2 + 2\sigma^2 r}$, then

$$\begin{aligned} \ldots &= \left(\frac{B}{S_0}\right)^{\frac{\alpha}{\sigma^2}}\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\ln^2\left(\frac{B}{S(0)}\right)+\beta^2 t^2}{2\sigma^2 t}\right\}dt \\ \ldots &= \left(\frac{B}{S_0}\right)^{\frac{\alpha}{\sigma^2}}\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\ln^2\left(\frac{B}{S(0)}\right)\pm 2\ln\left(\frac{B}{S(0)}\right)\beta t+\beta^2 t^2}{2\sigma^2 t}\right\}dt \\ &= \left(\frac{B}{S_0}\right)^{\frac{\alpha\pm\beta}{\sigma^2}}\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma t\sqrt{2\pi t}}\exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right)\pm\beta t\right)^2}{2\sigma^2 t}\right\}dt \end{aligned}$$

We note that

$$\frac{\ln\left(\frac{B}{S(0)}\right)}{\sigma\sqrt{t}} = \frac{\ln\left(\frac{B}{S(0)}\right)-\beta t}{2\sigma\sqrt{t}} + \frac{\ln\left(\frac{B}{S(0)}\right)+\beta t}{2\sigma\sqrt{t}} \tag{8}$$

and can thus split up the integral into

$$
\begin{aligned}
\ldots \quad = \quad & \left(\frac{B}{S_0}\right)^{\frac{\alpha\pm\beta}{\sigma^2}} \left[\int_0^T \frac{\ln\left(\frac{B}{S(0)}\right) - \beta t}{2\sigma t\sqrt{2\pi t}} \exp\left\{ -\frac{\left(\ln\left(\frac{B}{S(0)}\right) \pm \beta t\right)^2}{2\sigma^2 t} \right\} dt \right. \\
& \left. + \int_0^T \frac{\ln\left(\frac{B}{S(0)}\right) + \beta t}{2\sigma t\sqrt{2\pi t}} \exp\left\{ -\frac{\left(\ln\left(\frac{B}{S(0)}\right) \pm \beta t\right)^2}{2\sigma^2 t} \right\} dt \right]
\end{aligned}
$$

We now evaluate these integral separately. For both integrals, we apply a change of variable from t to $x_\pm$ by setting

$$
x_\pm(t) = \frac{\ln\left(\frac{B}{S(0)}\right) \pm \beta t}{\sigma\sqrt{t}}, \qquad \frac{dx_\pm}{dt} = -\frac{\ln\left(\frac{B}{S(0)}\right) \mp \beta t}{2\sigma t\sqrt{t}}
$$

This suggests, to use the positive version of the first integral and change the variable to x_+ and the negative version of the second integral in combination with change of variable to x_-. This approach eliminates all terms depending on t inside the integrals. We further note that $\lim_{x_\pm \downarrow 0} = \infty$. The first integral becomes

$$
\begin{aligned}
& \left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \int_0^T \frac{\ln\left(\frac{B}{S(0)}\right) - \beta t}{2\sigma t\sqrt{2\pi t}} \exp\left\{ -\frac{\left(\ln\left(\frac{B}{S(0)}\right) + \beta t\right)^2}{2\sigma^2 t} \right\} dt \\
& -\left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \int_\infty^{x_+(T)} \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x_+^2}{2} \right\} dx_+ \\
= \quad & \left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \mathcal{N}\left(-x_+(T)\right) \\
= \quad & \left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \mathcal{N}\left(\frac{\ln\left(\frac{S(0)}{B}\right) - \beta T}{\sigma\sqrt{T}} \right)
\end{aligned}
$$

The second integral becomes

$$
\begin{aligned}
&\left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \int_0^T \frac{\ln\left(\frac{B}{S(0)}\right) + \beta t}{2\sigma t\sqrt{2\pi t}} \exp\left\{-\frac{\left(\ln\left(\frac{B}{S(0)}\right) - \beta t\right)^2}{2\sigma^2 t}\right\} dt \\
&-\left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \int_\infty^{x_+(T)} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x_-^2}{2}\right\} dx_- \\
= \quad &\left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \mathcal{N}\left(-x_-(T)\right) \\
= \quad &\left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \mathcal{N}\left(\frac{\ln\left(\frac{S(0)}{B}\right) + \beta T}{\sigma\sqrt{T}}\right)
\end{aligned}
$$

Thus, the value of the American digital call is

$$
V(0) = \left(\frac{B}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \mathcal{N}\left(\frac{\ln\left(\frac{S(0)}{B}\right) - \beta T}{\sigma\sqrt{T}}\right) + \left(\frac{B}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \mathcal{N}\left(\frac{\ln\left(\frac{S(0)}{B}\right) + \beta T}{\sigma\sqrt{T}}\right)
$$