Derivation of the Local Volatility as a Function of the Implied Volatility

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This document closely follows Jim Gatheral's book "The Volatility Surface - A Practitioner's Guide" (Wiley, 2006). It provides all missing intermediate steps in the derivation of the local volatility in terms of the Black-Scholes implied volatility that are left out in the original text on pages 11 - 13.

Let $C_{BS}(K, T, \sigma_{BS})$ be the price of a European call option in the Black model where

$$F_{0,T} = S_0 \exp\left\{\int_0^T (r_t - d_t)\, dt\right\}$$

is the current time T forward price of the underlying asset. The option price is then given by

$$C_{BS}(K, T, \sigma_{BS}) = F_{0,T}\Phi(d_+) - K\Phi(d_-)$$

where

$$d_\pm = \frac{\left(\frac{F_{0,T}}{K}\right) \pm \frac{\sigma^2}{2}}{\sigma\sqrt{T}}$$

We first apply a change of coordinates from (K, T) to (y, T), where $y = \ln\left(\frac{K}{F_{0,T}}\right)$, and get $(K, T) = (F_{0,T}e^y, T)$. The partial derivatives of $C(K, T) = C(F_{0,T}e^y, T)$ then become

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$$
\begin{aligned}
\frac{\partial C(K,T)}{\partial K} &= \frac{\partial C(y,T)}{\partial y}\frac{\partial y}{\partial K} = \frac{1}{K}\frac{\partial C(y,T)}{\partial y} \\
\frac{\partial^2 C(K,T)}{\partial K^2} &= \frac{1}{K^2}\left(\frac{\partial^2 C(y,T)}{\partial y^2} - \frac{\partial C(y,T)}{\partial y}\right) \\
\frac{\partial C(K,T)}{\partial T} &= \frac{\partial C(y,T)}{\partial T} + \frac{\partial C(y,T)}{\partial y}\frac{\partial y}{\partial T} \\
&= \frac{\partial C(y,T)}{\partial T} - (r(T) - d(T))\frac{\partial C(y,T)}{\partial y}
\end{aligned}
$$

In the new coordinates (y, T), the Dupire equation becomes

$$
\frac{\partial C(y,T)}{\partial T} = \frac{\sigma^2(K,T)}{2}\left(\frac{\partial^2 C(y,T)}{\partial y^2} - \frac{\partial C(y,T)}{\partial y}\right) + (r_T - d_T)\, C(y,T) \tag{1}
$$

Let $\sigma_{BS}(K,T)$ be the implied volatility surface that matches the market prices of the continuum of call options, i.e.

$$
C(K,T) = C_{BS}\left(K, T, \sigma_{BS}(K,T)\right) \quad \forall K, T \in \mathbb{R}_+
$$

Another change of coordinates from (y, T) to (y, w), where $w = \sigma^2_{BS}(K,T)T$ is the total Black-Scholes implied variance, yields the call price $C(K,T) = C\left(F_{0,T}e^y, \frac{w}{\sigma^2_{BS}}\right)$ as a function of two dimensionless variables. The Black formula becomes

$$
C_{BS}(y,w) = F_{0,T}\left[\Phi\left(-\frac{y}{\sqrt{w}} + \frac{\sqrt{w}}{2}\right) - e^y\Phi\left(-\frac{y}{\sqrt{w}} - \frac{\sqrt{w}}{2}\right)\right]
$$

In order to transform the Dupire Equation (1) to the coordinates (y, w), we first compute

$$
\begin{aligned}
\frac{\partial C(y,T)}{\partial y} &= \frac{\partial C(y,w)}{\partial y} + \frac{\partial C(y,w)}{\partial w}\frac{\partial w}{\partial y} \\
\frac{\partial^2 C(y,T)}{\partial y^2} &= \frac{\partial^2 C(y,w)}{\partial y^2} + 2\frac{\partial^2 C(y,w)}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial^2 C(y,w)}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial C(y,w)}{\partial w}\frac{\partial^2 w}{\partial y^2} \\
\frac{\partial C(y,T)}{\partial T} &= \frac{\partial C(y,w)}{\partial T} + \frac{\partial C(y,w)}{\partial w}\frac{\partial w}{\partial T}
\end{aligned}
$$

In the new coordinates, the transformed Dupire equation becomes

$$
\begin{aligned}
& \frac{\partial C(y,w)}{\partial T} + \frac{\partial C(y,w)}{\partial w}\frac{\partial w}{\partial T} \\
= \; & \frac{\sigma^2(K,T)}{2}\left(\frac{\partial^2 C(y,w)}{\partial y^2} + 2\frac{\partial^2 C(y,w)}{\partial y \partial w}\frac{\partial w}{\partial y} + \frac{\partial^2 C(y,w)}{\partial w^2}\left(\frac{\partial w}{\partial y}\right)^2\right. \\
& \left. + \frac{\partial C(y,w)}{\partial w}\frac{\partial^2 w}{\partial y^2} - \frac{\partial C(y,w)}{\partial y} - \frac{\partial C(y,w)}{\partial w}\frac{\partial w}{\partial y}\right) + (r_T - d_T)\, C(y,w) \qquad (2)
\end{aligned}
$$

Since the call value $C_{BS}(y,w)$ satisfies this equation as well, we proceed by explicitely computing the required partial derivatives

$$
\begin{aligned}
\frac{\partial C_{BS}(y,w)}{\partial w} &= \frac{\partial C_{BS}(y,w)}{\partial \sigma}\frac{\partial \sigma}{\partial w} = \frac{F_{0,T}\phi(d_+)}{2\sqrt{w}} \\
\frac{\partial^2 C_{BS}(y,w)}{\partial w^2} &= F_{0,T}\phi(d_+)\left[-\frac{1}{4\sqrt{w^3}} - \frac{1}{2\sqrt{w}}d_+\frac{\partial d_+}{\partial w}\right] \\
&= \frac{F_{0,T}\phi(d_+)}{2\sqrt{w}}\left[-\frac{1}{2w^2} + \left(\frac{y}{\sqrt{w}} - \frac{\sqrt{w}}{2}\right)\left(\frac{y}{2\sqrt{w^3}} + \frac{1}{4\sqrt{w}}\right)\right] \\
&= \frac{\partial C_{BS}(y,w)}{\partial w}\left[-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C_{BS}(y,w)}{\partial y} &= \frac{\partial C_{BS}(y,w)}{\partial K}\frac{\partial K}{\partial y} = -F_{0,T}e^y\Phi(d_-) \\
\frac{\partial^2 C_{BS}(y,w)}{\partial y^2} &= -F_{0,T}e^y\Phi(d_-) - F_{0,T}\phi(d_-)\frac{\partial d_-}{\partial y} \\
&= -F_{0,T}e^y\Phi(d_-) + \frac{F_{0,T}\phi(d_+)}{\sqrt{w}} \\
\frac{\partial^2 C_{BS}(y,w)}{\partial y^2} - \frac{\partial C_{BS}(y,w)}{\partial y} &= \frac{F_{0,T}\phi(d_+)}{\sqrt{w}} = 2\frac{\partial C_{BS}(y,w)}{\partial w}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 C_{BS}(y,w)}{\partial y \partial w} &= -F_{0,T}e^y\phi(d_-)\frac{\partial d_-}{\partial w} \\
&= -F_{0,T}\phi(d_+)\left(\frac{y}{2\sqrt{w^3}} - \frac{1}{4\sqrt{w}}\right) \\
&= \frac{\partial C_{BS}(y,w)}{\partial w}\left(\frac{1}{2} - \frac{y}{w}\right) \\
\frac{\partial C_{BS}(y,w)}{\partial T} &= \frac{\partial F_{0,T}}{\partial T}\left[\Phi\left(-\frac{y}{\sqrt{w}} + \frac{\sqrt{w}}{2}\right) - e^y\Phi\left(-\frac{y}{\sqrt{w}} - \frac{\sqrt{w}}{2}\right)\right] \\
&= (r_T - d_T)\, C_{BS}(y,w)
\end{aligned}
$$

When substituting these results back into the transformed Dupire Equation (2), the $(r_T - d_T)\, C_{BS}(y,w)$ terms cancel out on both sides and we get

$$
\begin{aligned}
\frac{\partial C(y,w)}{\partial w}\frac{\partial w}{\partial T} \quad &= \quad \frac{\sigma^2(K,T)}{2}\frac{\partial C_{BS}(y,w)}{\partial w}\left[2 + 2\left(\frac{1}{2} - \frac{y}{w}\right)\frac{\partial w}{\partial y}\right. \\
&\quad \left. + \left(-\frac{1}{8} - \frac{1}{2w} + \frac{y^2}{2w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial y}\right]
\end{aligned}
$$

Cancelling out the factor $\frac{\partial C_{BS}(y,w)}{\partial w}$ and simplifying yields

$$
\frac{\partial w}{\partial T} = \sigma^2(K,T)\left[1 - \frac{y}{w}\frac{\partial w}{\partial y} + \frac{1}{4}\left(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}\right] \tag{3}
$$

Equation (3) can be easily inverted to obtain a closed form solution for the local variance in terms of the Black-Scholes implied volatility surface. For the special case when the volatility smile is flat for each maturity, the partial derivatives with respect to the log-moneyness y vanish and the local variance simplifies to

$$
\sigma^2(T) = \frac{\partial w}{\partial T}
$$