Stochastic Calculus for Finance II
-
some Solutions to Chapter III

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Exercise 3.1

We first note that for $u_1 < u_2$, the Brownian increment $W(u_2) - W(u_1)$ is independent of the σ-algebra $\mathcal{F}(u_1)$ by Definition 3.3.3(iii). By Definition 2.2.3, the random variable $X = W(u_2) - W(u_1)$ is independent of the σ-algebra $\mathcal{F}(u_1)$ if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

for all $A \in \sigma(X)$ and $B \in \mathcal{F}(u_1)$. By Definition 3.3.3(i) information accumulates and every set in $\mathcal{F}(t)$ for $t < u_1$ is also in $\mathcal{F}(u_1)$. Thus, we have

$$\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$$

for all $A \in \sigma(X)$ and $C \in \mathcal{F}(t)$ and it follows that the increment $W(u_2) - W(u_1)$ is independent of $\mathcal{F}(t)$.

Exercise 3.2

$$\begin{aligned}
\mathbb{E}\left[W^2(t) - t \middle| \mathcal{F}(s)\right] &= \mathbb{E}\left[(W(t) - W(s))^2 + 2W(t)W(s) - W^2(s) - t \middle| \mathcal{F}(s)\right] \\
&= \mathbb{E}\left[(W(t) - W(s))^2\right] + 2W(s)\mathbb{E}\left[W(t) | \mathcal{F}(s)\right] - W^2(s) - t \\
&= \mathrm{Var}\left(W(t) - W(s)\right) + 2W^2(s) - W^2(s) - t \\
&= t - s + W^2(s) - t \\
&= W^2(s) - s \qquad \text{(q.e.d.)}
\end{aligned}$$

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In the second step, we used that the Brownian increment $W(t)-W(s)$ is independent of $\mathcal{F}(s)$ by Definition 3.3.3(iii) and that by Theorem 2.2.5 any function of this increment is also independent of the σ-algebra. Furthermore, $W(s)$ is $\mathcal{F}(s)$-measurable and can thus be taken outside the conditional expectation.

In the third step, we used that the expected value of the Brownian increment is zero by Definition 3.3.1 to obtain the first term and the martingale property of Brownian motion from Theorem 3.3.4 to get the second term.

Exercise 3.3 (Normal kurtosis)

The third and fourth derivative are

$$\begin{aligned}
\varphi'''(u) &= \mathbb{E}\left[(X-\mu)^3 e^{u(X-\mu)}\right] = \left(3u\sigma^4 + u^3\sigma^6\right) e^{\frac{1}{2}\sigma^2 u^2} \\
\varphi''''(u) &= \mathbb{E}\left[(X-\mu)^4 e^{u(X-\mu)}\right] = \left(3\sigma^4 + 6u^2\sigma^6 + u^4\sigma^8\right) e^{\frac{1}{2}\sigma^2 u^2}
\end{aligned}$$

and it follows that

$$\mathbb{E}\left[(X-\mu)^4\right] = \varphi''''(0) = 3\sigma^4 \qquad \text{(q.e.d.)}.$$

Exercise 3.4 (Other variations of Brownian motion)

(i) As by the hint, we have

$$\sum_{j=1}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2 \leq \max_{0\leq k\leq n-1} \left|W(t_{k+1}) - W(t_k)\right| \cdot \sum_{j=0}^{n-1} \left|W(t_{j+1}) - W(t_j)\right|.$$

In the limit as the number of partition points increases, the left hand side converges almost surely to the quadratic variation

$$\mathbb{P}\left(\lim_{||\Pi||\to 0} \sum_{j=1}^{n-1} \left(W(t_{j+1}) - W(t_j)\right)^2 = T\right) = 1.$$

The first term on the right hand side converges almost surely to zero since Brownian motion has continuous sample paths almost surely by Theorem 3.3.2, i.e.

$$\mathbb{P}\left(\lim_{||\Pi||\to 0} \max_{0\leq k\leq n-1} \left|W(t_{k+1}) - W(t_k)\right| = 0\right) = 1.$$

Rearranging the inequality gives

$$\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| \geq \frac{\max_{0\leq k\leq n-1} |W(t_{k+1}) - W(t_k)|}{\sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|}.$$

Now, since the numerator converges almost surely to a positive constant and the denominator converges almost surely to zero, it follows that the fraction converges almost surely to plus infinity. Since it is bounded from above by the left hand side, it follows that

$$\mathbb{P}\left(\lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)| = \infty \right) = 1.$$

(ii) Similar to (i), the sample cubic variation can be bounded by

$$0 \leq \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 \leq \max_{0\leq k\leq n-1} |W(t_{k+1}) - W(t_k)| \cdot \sum_{j=0}^{n-1} (W(t_{j+1}) - W(t_j))^2 .$$

As argued in (i), the first term on the right hand side converges almost surely to zero and the second term converges almost surely to the quadratic variation. Thus, the right hand side converges almost surely to zero and consequently the sample cubic variation converges almost surely to zero as well, i.e.

$$\mathbb{P}\left(\lim_{||\Pi||\to 0} \sum_{j=0}^{n-1} |W(t_{j+1}) - W(t_j)|^3 = 0 \right) = 1.$$

Exercise 3.5 (Black-Scholes-Merton formula)

Since $W(T)$ is known to be $\mathcal{N}(0,T)$ normally distributed or equivalently $\frac{W(T)}{\sqrt{T}}$ is $\mathcal{N}(0,1)$ standard normally distributed, Theorem 1.5.2 allows us to compute the expectation as

$$\mathbb{E}\left[e^{-rT}(S(T)-K)^+\right] = e^{-rT}\int_{-\infty}^{\infty}\left(S(0)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T + \sigma\sqrt{T}z\right\} - K\right)^+ \mathcal{N}'(z)dz$$

where

$$\mathcal{N}'(z) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}$$

denotes the standard normal density function. Next, we want to eliminate the max function inside the integral. We observe that the terminal payoff of the call option is non-zero if $S(T) > X$ or equivalently

$$z \geq \frac{\ln\left(\frac{K}{S(0)}\right) - \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = -d_-.$$

Changing the lower limit of integration from $-\infty$ to $-d_-$ allows us to drop the max function and we get

$$\begin{aligned} \ldots &= e^{-rT}\int_{-d_-}^{-\infty}\left(S(0)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z\right\}-K\right)\mathcal{N}'(z)dz \\ &= e^{-rT}\left[\int_{-d_-}^{\infty}S(0)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z\right\}\mathcal{N}'(z)dz-\int_{-d_-}^{\infty}K\mathcal{N}'(z)dz\right]. \end{aligned}$$

The second integral evaluates to

$$\begin{aligned} \int_{-d_-}^{\infty}K\mathcal{N}'(z)dz &= K\int_{-d_-}^{\infty}\mathcal{N}'(z)dz \\ &= K\mathbb{P}\left\{z \geq -d_-\right\} \\ &= K\mathbb{P}\left\{z \leq d_-\right\} \\ &= K\mathcal{N}\left(d_-\right). \end{aligned}$$

Here, we exploited the symmetry of the normal distribution in the third step. Using the definition of the standard normal density, we can write the first integral as

$$\begin{aligned} &\int_{-d_-}^{\infty}S(0)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z\right\}\mathcal{N}'(z)dz \\ =\ &\int_{-d_-}^{\infty}S(0)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}z\right\}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2}{2}\right\}dz \\ =\ &S(0)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T\right\}\int_{-d_-}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2-2\sigma\sqrt{T}z}{2}\right\}dz \\ =\ &S(0)\exp\left\{\left(r-\frac{1}{2}\sigma^2\right)T\right\}\int_{-d_-}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2-2\sigma\sqrt{T}z\pm\sigma^2T}{2}\right\}dz \\ =\ &S(0)e^{rT}\int_{-d_-}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{z^2-2\sigma\sqrt{T}z+\sigma^2T}{2}\right\}dz \\ =\ &S(0)e^{rT}\int_{-d_-}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\left(z-\sigma\sqrt{T}\right)^2}{2}\right\}dz. \end{aligned}$$

We now make a change of variable by defining $x = z - \sigma\sqrt{T}$ and get

$$\begin{aligned} \dots &= S(0)e^{rT} \int_{-d_- - \sigma\sqrt{T}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx \\ &= S(0)e^{rT} \int_{-d_- - \sigma\sqrt{T}}^{\infty} \mathbb{P}\left\{\epsilon = x\right\} dx \\ &= S(0)e^{rT}\mathbb{P}\left\{\epsilon \geq -d_- - \sigma\sqrt{T}\right\} \\ &= S(0)e^{rT}\mathbb{P}\left\{\epsilon \leq d_- + \sigma\sqrt{T}\right\} \\ &= S(0)e^{rT}\mathcal{N}\left(d_- + \sigma\sqrt{T}\right). \end{aligned}$$

Defining $d_+ = d_- + \sigma\sqrt{T}$ and combining all previous results yields

$$\mathbb{E}\left[e^{-rT}\left(S(T) - K\right)^+\right] = S(0)\mathcal{N}\left(d_+\right) - Ke^{-rT}\mathcal{N}\left(d_-\right).$$

Exercise 3.6

(i) We can rewrite the expectation as

$$\mathbb{E}\left[f(X(t))\middle|\mathcal{F}(s)\right] = \mathbb{E}\left[f(X(s) + X(t) - X(s))\middle|\mathcal{F}(s)\right].$$

Note that $X(s) = \mu s + W(s)$ is $\mathcal{F}(s)$-measurable while the increment $X(t) - X(s) = \mu(t-s) + W(t) - W(s)$ is independent of $\mathcal{F}(s)$. By Lemma 2.3.4, we have

$$\mathbb{E}\left[f(X(s) + X(t) - X(s))\middle|\mathcal{F}(s)\right] = g(X(s)),$$

where

$$g(x) = \mathbb{E}\left[f(x + X(t) - X(s))\right].$$

Since $X(t) - X(s)$ is normally distributed with mean $\mu(t-s)$ and variance $t-s$, we can compute this expectation by Theorem 1.5.2 via

$$g(x) = \int_{-\infty}^{\infty} f(x+z)\frac{1}{\sqrt{2\pi(t-s)}}\exp\left\{-\frac{\left(z-\mu(t-s)\right)^2}{2(t-s)}\right\} dz.$$

Making a change of variable by defining $y = x + z$ and rearranging yields the desired result

$$g(x) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{-\infty}^{\infty} f(y) \exp\left\{ -\frac{(y - x - \mu(t-s))^2}{2(t-s)} \right\} dy.$$

(ii) Analogous to (i), we can rewrite the expectation as

$$\mathbb{E}\left[f(S(t)) \middle| \mathcal{F}(s) \right] = \mathbb{E}\left[f\left(S(s) \cdot \frac{S(t)}{S(s)} \right) \middle| \mathcal{F}(s) \right],$$

where $S(s) = S(0) \exp\{\sigma W(s) + \nu s\}$ is $\mathcal{F}(s)$-measurable and the increment $\frac{S(t)}{S(s)} = \exp\{\sigma(W(t) - W(s)) + \nu(t-s)\}$ is independent of $\mathcal{F}(s)$. By Lemma 2.3.4, we have

$$\mathbb{E}\left[f\left(S(s) \cdot \frac{S(t)}{S(s)} \right) \middle| \mathcal{F}(s) \right] = g(S(s)),$$

where

$$g(x) = \mathbb{E}\left[f\left(x \cdot \frac{S(t)}{S(s)} \right) \right].$$

Since $\sigma(W(t) - W(s)) + \nu(t-s)$ is normally distributed with mean $\nu(t-s)$ and variance $\sigma^2(t-s)$, it follows that the ratio $\frac{S(t)}{S(s)}$ is log-normally distributed with the same parameters. The expectation can be computed as

$$g(x) = \int_0^{\infty} f(x \cdot z) \frac{1}{z\sqrt{2\pi\sigma^2(t-s)}} \exp\left\{ -\frac{\ln z - \nu(t-s)}{2\sigma^2(t-s)} \right\} dz.$$

Making a change of variable by defining $y = x \cdot z$ with $dy = x dz$ yield the desired result

$$g(x) = \int_0^{\infty} f(y) \frac{1}{y\sqrt{2\pi\sigma^2(t-s)}} \exp\left\{ -\frac{\ln\left(\frac{y}{x}\right) - \nu(t-s)}{2\sigma^2(t-s)} \right\} dy.$$

Exercise 3.7

(i) Substituting for $X(t)$ gives

$$Z(t) = \exp\left\{\sigma W(t) - \frac{1}{2}\sigma^2 t\right\}$$

and by Theorem 3.6.1, $Z(t)$ is an exponential martingale.

(ii) Since $Z(t)$ is a martingale, the stopped process $Z(t \wedge \tau_m)$ is also a martingale and we have that

$$1 = Z(0) = \mathbb{E}\left[Z(t \wedge \tau_m)\right] = \mathbb{E}\left[\exp\left\{\sigma X(t \wedge \tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\}\right]$$

for $t \geq 0$.

(iii) We closely follow the argument in Section 3.6. For any time $t \leq \tau_m$, the drifted Brownian motion is at or below the level m and thus we have that for all $t \geq 0$,

$$0 \leq \exp\left\{\sigma X(t \wedge \tau_m)\right\} \leq e^{\sigma m}.$$

Furthermore, since we assume that $\mu \geq 0$ and $\sigma > 0$, the term $\sigma\mu + 0.5\sigma^2$ is strictly positive and thus

$$\lim_{t\to\infty} \exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\} = \mathbb{I}_{\{\tau_m < \infty\}} \exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}.$$

Now, since the first term is bounded and the second converges to zero when $\tau_m = \infty$, it follows that the whole expression converges to zero in this case. Thus,

$$\lim_{t\to\infty} \exp\left\{\sigma X(t \wedge \tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\} = \mathbb{I}_{\{\tau_m < \infty\}} \exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}.$$

We now take the limit for $t \to \infty$ inside the martingale equation obtained in (ii). The interchange of limit and expectation is justified by Theorem 1.4.9 as we can define a non-negative random variable (constant) $Y = e^{\sigma m} < \infty$ such that

$$\mathbb{P}\left(Y \geq \left|\exp\left\{\sigma X(t \wedge \tau_m) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)(t \wedge \tau_m)\right\}\right|\right) = 1$$

as argued before. Thus,

$$
\begin{aligned}
1 &= \lim_{t\to\infty} \mathbb{E}\left[\exp\left\{\sigma X\left(t\wedge\tau_m\right) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\left(t\wedge\tau_m\right)\right\}\right] \\
&= \mathbb{E}\left[\lim_{t\to\infty} \exp\left\{\sigma X\left(t\wedge\tau_m\right) - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\left(t\wedge\tau_m\right)\right\}\right] \\
&= \mathbb{E}\left[\mathbb{I}_{\{\tau_m<\infty\}} \exp\left\{\sigma m - \left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right]
\end{aligned}
$$

or

$$
\mathbb{E}\left[\mathbb{I}_{\{\tau_m<\infty\}} \exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right] = e^{-\sigma m}.
$$

Taking the limit for $\sigma \to 0$ yields

$$
\mathbb{E}\left[\mathbb{I}_{\{\tau_m<\infty\}}\right] = \mathbb{P}\left\{\tau_m < \infty\right\} = 1.
$$

Since the stopping time τ_m is almost surely finite, we can drop the conditioning to obtain

$$
\mathbb{E}\left[\exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right] = e^{-\sigma m}.
$$

We define

$$
\alpha = \sigma\mu + \frac{1}{2}\sigma^2 \quad \Leftrightarrow \quad \sigma_{\pm} = -\mu \pm \sqrt{2\alpha + \mu^2}.
$$

The condition $\sigma > 0$ is only satisfied by the positive root and we obtain the Laplace transform

$$
\mathbb{E}\left[e^{-\alpha\tau_m}\right] = \exp\left\{m\mu - m\sqrt{2\alpha + \mu^2}\right\}.
$$

(iv) Differentiating the Laplace transform w.r.t. to α yields

$$
\mathbb{E}\left[\tau_m e^{-\alpha\tau_m}\right] = \frac{m}{\sqrt{2\alpha + \mu^2}} \exp\left\{m\mu - m\sqrt{2\alpha + \mu^2}\right\},
$$

which in the limit for $\alpha \to 0$ becomes $\mathbb{E}[\tau_m] = \infty$. Note that the first term on the right hand side converges to infinity since we assume that $m > 0$ while the second term converges to a constant.

(v) If $\mu < 0$ and $\sigma > 0$, then the term $\sigma\mu + 0.5\sigma^2$ is still strictly positive and our analysis in (iii) up to the equation

$$\mathbb{E}\left[\mathbb{I}_{\{\tau_m<\infty\}} \exp\left\{-\left(\sigma\mu + \frac{1}{2}\sigma^2\right)\tau_m\right\}\right] = e^{-\sigma m}$$

still holds. We now take the limit for $\sigma \to -2\mu = 2|\mu|$ (since $\mu < 0$) such that the exponential term inside the expectation converges to one and obtain

$$\mathbb{E}\left[\mathbb{I}_{\{\tau_m<\infty\}}\right] = \mathbb{P}\left\{\tau_m < \infty\right\} = e^{-2m|\mu|} < 1.$$

Note that there is a typo in the exercise (at least in the 2004 edition) - instead of $\mathbb{P}\{\tau_m < \infty\} = e^{-2x|\mu|}$ it should read $\mathbb{P}\{\tau_m < \infty\} = e^{-2m|\mu|}$ which is what we derived above.

In contrast to (iii), τ_m is infinite with non-zero probability and we cannot simply drop the conditioning. Defining α in the same way as in (iii), again taking the positive root for σ gives

$$\mathbb{E}\left[\mathbb{I}_{\{\tau_m<\infty\}} e^{-\alpha m}\right] = \exp\left\{m\mu - m\sqrt{2\alpha + \mu^2}\right\}.$$

However, since $e^{-\alpha m} = 0$ if and only if $\tau_m = \infty$ since $m > 0$ the conditioning can still be dropped and we obtain the same result as in (iii).

Exercise 3.8

(i) We have

$$
\begin{aligned}
\varphi_n(u) &= \mathbb{E}\left[\exp\left\{\frac{u}{\sqrt{n}} M_{nt,n}\right\}\right] \\
&= \mathbb{E}\left[\exp\left\{\frac{u}{\sqrt{n}} \sum_{k=1}^{nt} X_{k,n}\right\}\right] \\
&= \mathbb{E}\left[\prod_{k=1}^{nt} \exp\left\{\frac{u}{\sqrt{n}} X_{k,n}\right\}\right] \\
&= \prod_{k=1}^{nt} \mathbb{E}\left[\exp\left\{\frac{u}{\sqrt{n}} X_{k,n}\right\}\right] \\
&= \left(\mathbb{E}\left[\exp\left\{\frac{u}{\sqrt{n}} X_{1,n}\right\}\right]\right)^{nt} \\
&= \left(e^{\frac{u}{\sqrt{n}}} \tilde{p}_n + e^{-\frac{u}{\sqrt{n}}} \tilde{q}_n\right)^{nt} \\
&= \left(e^{\frac{u}{\sqrt{n}}} \frac{\frac{r}{n} + 1 - e^{-\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}} - e^{-\frac{u}{\sqrt{n}}} \frac{\frac{r}{n} + 1 - e^{\sigma/\sqrt{n}}}{e^{\sigma/\sqrt{n}} - e^{-\sigma/\sqrt{n}}}\right)^{nt} \qquad \text{(q.e.d.).}
\end{aligned}
$$

Here, we used that the increments $X_{1,n}, \ldots, X_{n,n}$ are independent in the fourth equality to write the expectation as the product as the product of the expectations. The fifth step uses that increments are also identically distributed such that it is sufficient to compute the expression for $X_{1,n}$.

(ii) First note that

$$
\varphi_{\frac{1}{x^2}}(u) = \left(e^{ux} \frac{rx^2 + 1 - e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}} - e^{-ux} \frac{rx^2 + 1 - e^{\sigma x}}{e^{\sigma x} - e^{-\sigma x}}\right)^{\frac{t}{x^2}}.
$$

Thus

$$
\begin{aligned}
\ln \varphi_{\frac{1}{x^2}}(u) &= \frac{t}{x^2} \ln\left(e^{ux} \frac{rx^2 + 1 - e^{-\sigma x}}{e^{\sigma x} - e^{-\sigma x}} - e^{-ux} \frac{rx^2 + 1 - e^{\sigma x}}{e^{\sigma x} - e^{-\sigma x}}\right) \\
&= \frac{t}{x^2} \ln\left(\left(rx^2 + 1\right) \frac{e^{ux} - e^{-ux}}{e^{\sigma x} - e^{-\sigma x}} + \frac{e^{(\sigma - u)x} - e^{-(\sigma - u)x}}{e^{\sigma x} - e^{-\sigma x}}\right) \\
&= \frac{t}{x^2} \ln\left(\frac{(rx^2 + 1)\sinh(ux) + \sinh((\sigma - u)x)}{\sinh(\sigma x)}\right) \\
&= \frac{t}{x^2} \ln\left(\frac{(rx^2 + 1)\sinh(ux) + \sinh(\sigma x)\cosh(ux) - \cosh(\sigma x)\sinh(ux)}{\sinh(\sigma x)}\right) \\
&= \frac{t}{x^2} \ln\left(\cosh(ux) + \frac{(rx^2 + 1 - \cosh(\sigma x))\sinh(ux)}{\sinh(\sigma x)}\right) \qquad \text{(q.e.d.).}
\end{aligned}
$$

(iii) Using the hint, we get

$$\begin{aligned}
& \cosh(ux) + \frac{(rx^2 + 1 - \cosh(\sigma x))\sinh(ux)}{\sinh(\sigma x)} \\
= & \ 1 + \frac{1}{2}u^2x^2 + \mathcal{O}\left(x^4\right) + \frac{\left(rx^2 - \frac{1}{2}\sigma^2x^2 + \mathcal{O}\left(x^2\right)\right)\left(ux + \mathcal{O}\left(x^3\right)\right)}{\sigma x + \mathcal{O}\left(x^3\right)} \\
= & \ 1 + \frac{1}{2}u^2x^2 + \mathcal{O}\left(x^4\right) + \frac{\left(r - \frac{1}{2}\sigma^2\right)ux^3\left(1 + \mathcal{O}\left(x^2\right)\right)}{\sigma x\left(1 + \mathcal{O}\left(x^2\right)\right)} \\
= & \ 1 + \frac{1}{2}u^2x^2 + \frac{rux^2}{\sigma} - \frac{1}{2}\sigma ux^2 + \mathcal{O}\left(x^4\right) \qquad \text{(q.e.d.)}.
\end{aligned}$$

Here, we used that

$$ux + \mathcal{O}\left(x^3\right) = ux\left(1 + \mathcal{O}\left(x^2\right)\right)$$

in the second equality.

(iv) Finally,

$$\begin{aligned}
\ln\varphi_{\frac{1}{x^2}}(u) &= \frac{t}{x^2}\ln\left(1 + \left(\frac{1}{2}u + \frac{r}{\sigma} - \frac{1}{2}\sigma\right)ux^2 + \mathcal{O}\left(x^4\right)\right) \\
&= \frac{t}{x^2}\left[\left(\frac{1}{2}u + \frac{r}{\sigma} - \frac{1}{2}\sigma\right)ux^2 + \mathcal{O}\left(x^4\right)\right] \\
&= \left(\frac{1}{2}u + \frac{r}{\sigma} - \frac{1}{2}\sigma\right)tu + \mathcal{O}\left(x^2\right).
\end{aligned}$$

Thus,

$$\lim_{x\downarrow 0}\ln\varphi_{\frac{1}{x^2}}(u) = \frac{1}{2}tu^2 + \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)tu.$$

It follows that

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}\left[\exp\left\{\frac{u\sigma}{\sqrt{n}}M_{nt,n}\right\}\right] &= \lim_{n\to\infty}\varphi_n(\sigma u) \\
&= \lim_{x\downarrow 0}\varphi_{\frac{1}{x^2}}(\sigma u) \\
&= \exp\left\{\lim_{x\downarrow 0}\ln\varphi_{\frac{1}{x^2}}(\sigma u)\right\} \\
&= \exp\left\{\frac{1}{2}u^2\sigma^2t + \left(r - \frac{1}{2}\sigma^2\right)tu\right\}.
\end{aligned}$$

We recognize this as the moment generating function of a normal random variable and conclude that

$$\lim_{n\to\infty} \frac{\sigma}{\sqrt{n}} M_{nt,n} \sim \mathcal{N}\left(\left(r-\frac{1}{2}\sigma^2\right)t, \sigma^2 t\right) \qquad \text{(q.e.d.)}.$$

Exercise 3.9 (Laplace transform of first passage density)

(i) We first note that

$$\begin{aligned}
\frac{da_k(m)}{dm} &= -\frac{m}{\sqrt{2\pi}} \int_0^\infty t^{-k/2-1} \exp\left\{-\alpha t - \frac{m^2}{2t}\right\} dt \\
&= -\frac{m}{\sqrt{2\pi}} \int_0^\infty t^{-(k+2)/2} \exp\left\{-\alpha t - \frac{m^2}{2t}\right\} dt \\
&= -m a_{k+2}(m).
\end{aligned}$$

Thus

$$\begin{aligned}
g_m(\alpha, m) &= a_3(m) + m\frac{da_3(m)}{dm} \\
&= a_3(m) - m^2 a_5(m) \\
g_{mm}(\alpha, m) &= \frac{da_3(m)}{dm} - 2m a_5(m) - m^2 \frac{da_5(m)}{dm} \\
&= -3m a_5(m) + m^3 a_7(m).
\end{aligned}$$

(ii)

$$\begin{aligned}
a_5(m) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty t^{-5/2} \exp\left\{-\alpha t - \frac{m^2}{2t}\right\} dt \\
&= \qquad (1)
\end{aligned}$$