Stochastic Calculus for Finance II
-
some Solutions to Chapter IV

Matthias Thul*

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Exercise 4.1

This proof is fully analogous to the one of Theorem 4.2.1. We want to show that for $0 \leq s \leq t \leq T$

$$\mathbb{E}[I(t)|\mathcal{F}(s)] = I(s).$$

Assume again, that the $s \in [t_l, t_{l+1})$ and $t \in [t_k, t_{k+1})$ for $l \leq k$. We start by splitting up the sum into an $\mathcal{F}(s)$ measurable part and a part independent of $\mathcal{F}(s)$ and obtain

$$\begin{aligned} I(t) &= \sum_{j=0}^{l-1} \Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] + \Delta(t_l)\left[M(t_{l+1}) - M(t_l)\right] \\ &\quad + \sum_{j=l+1}^{k-1} \Delta(t_j)\left[M(t_{j+1}) - M(t_j)\right] + \Delta(t_k)\left[M(t) - M(t_k)\right] \end{aligned}$$

The first sum is $\mathcal{F}(s)$ measurable. The conditional expectation of the second term is

$$\begin{aligned} \mathbb{E}\left[\Delta(t_l)\left[M(t_{l+1}) - M(t_l)\right]\middle|\, \mathcal{F}(s)\right] &= \Delta(t_l)\left[\mathbb{E}\left[M(t_{l+1})\middle|\, \mathcal{F}(s)\right] - M(t_l)\right] \\ &= \Delta(t_l)\left[M(s) - M(t_l)\right]. \end{aligned}$$

Here we have used that $M(t_l)$ is $\mathcal{F}(s)$ measurable. Furthermore, the martingale property of $M(t)$ implied that $\mathbb{E}\left[M(t_{l+1})\middle|\, \mathcal{F}(s)\right] = M(s)$. The conditional expectation of each element of the second sum is

*The author can be contacted via <<firstname>>.<<lastname>>@gmail.com and http://www.matthiasthul.com.

$$
\begin{aligned}
\mathbb{E}\left[\Delta\left(t_j\right)\left[M\left(t_{j+1}\right)-M\left(t_j\right)\right]\mid\mathcal{F}(s)\right] &= \mathbb{E}\left[\mathbb{E}\left[\Delta\left(t_j\right)\left[M\left(t_{j+1}\right)-M\left(t_j\right)\right]\mid\mathcal{F}\left(t_j\right)\right]\mid\mathcal{F}(s)\right] \\
&= \mathbb{E}\left[\Delta\left(t_j\right)\left[\mathbb{E}\left[M\left(t_{j+1}\right)\mid\mathcal{F}\left(t_j\right)\right]-M\left(t_j\right)\right]\mid\mathcal{F}(s)\right] \\
&= \mathbb{E}\left[\Delta\left(t_j\right)\left[M\left(t_j\right)-M\left(t_j\right)\right]\mid\mathcal{F}(s)\right] \\
&= 0.
\end{aligned}
$$

Here, we first used the tower law of iterated conditioning, then took out all $\mathcal{F}\left(t_j\right)$ measurable terms from the inner conditional expectation and finally used the martingale property of $M(t)$ again. The same steps can be used to show that

$$\mathbb{E}\left[\Delta\left(t_k\right)\left[M(t)-M\left(t_k\right)\right]\mid\mathcal{F}(s)\right]=0.$$

It follows that

$$
\begin{aligned}
\mathbb{E}\left[I(t)\mid\mathcal{F}(s)\right] &= \sum_{j=0}^{l-1}\Delta\left(t_j\right)\left[M\left(t_{j+1}\right)-M\left(t_j\right)\right]+\Delta\left(t_l\right)\left[M(s)-M\left(t_l\right)\right] \\
&= I(s) \qquad \text{(q.e.d.)}
\end{aligned}
$$

Exercise 4.4 (Stratonovich integral)

(i) Since the elements in the sum are independent, we start by computing the mean and the variance of $\left(W\left(t_j^*\right)-W\left(t_j\right)\right)^2$. This is completely analogous to the proof of Theorem 3.4.3.

$$
\begin{aligned}
\mathbb{E}\left[\left(W\left(t_j^*\right)-W\left(t_j\right)\right)^2\right] &= \operatorname{Var}\left[W\left(t_j^*\right)-W\left(t_j\right)\right]=t_j^*-t_j \\
\mathbb{E}\left[\left(W\left(t_j^*\right)-W\left(t_j\right)\right)^4\right] &= 3\mathbb{E}\left[\left(W\left(t_j^*\right)-W\left(t_j\right)\right)^2\right]^2=3\left(t_j^*-t_j\right)^2 \\
\operatorname{Var}\left[\left(W\left(t_j^*\right)-W\left(t_j\right)\right)^2\right] &= 3\left(t_j^*-t_j\right)^2-\left(t_j^*-t_j\right)^2=2\left(t_j^*-t_j\right)^2
\end{aligned}
$$

Here, we have used that the forth moment of a normal random variable is three times its variance squared. The mean and variance of $Q_{\Pi/2}$ are then given by

$$
\begin{aligned}
\mathbb{E}\left[Q_{\Pi/2}\right] &= \sum_{j=0}^{n-1}\mathbb{E}\left[\left(W\left(t_j^*\right)-W\left(t_j\right)\right)^2\right] \\
&= \sum_{j=0}^{n-1}t_j^*-t_j=\sum_{j=0}^{n-1}\frac{t_{j+1}-t_j}{2}=\frac{1}{2}T
\end{aligned}
$$

$$\begin{aligned}
\operatorname{Var}\left[Q_{\Pi/2}\right] &= \sum_{j=0}^{n-1} \operatorname{Var}\left[\left(W\left(t_j^*\right) - W\left(t_j\right)\right)^2\right] \\
&= \sum_{j=0}^{n-1} 2\left(t_j^* - t_j\right)^2 = \sum_{j=0}^{n-1} \frac{1}{2}\left(t_{j+1} - t_j\right)^2 \\
&\leq \frac{||\Pi||}{2} \sum_{j=0}^{n-1} \frac{1}{2}\left(t_{j+1} - t_j\right) \\
\lim_{||\Pi|| \to 0} \operatorname{Var}\left[Q_{\Pi/2}\right] &= 0 \cdot \frac{1}{2} T = 0 \qquad \text{(q.e.d.)}
\end{aligned}$$

Since the expected value of $Q_{\Pi/2}$ is the same for all partitions, but the variance as limit zero when the maximum partition size gets smaller, we can conclude that $Q_{\Pi/2}$ has limit $\frac{1}{2}T$.

(ii) We expand the sum in the definition of the Stratonovich integral and add the terms necessary to obtain an Itô integral over the partition $Q_{\Pi/2}$.

$$\begin{aligned}
&\int_0^T W(t) \circ dW(t) \\
&= \lim_{||\Pi|| \to 0} \sum_{j+0}^{n-1} W\left(t_j^*\right)\left(W\left(t_{j+1}\right) - W\left(t_j\right)\right) \\
&= \lim_{||\Pi|| \to 0} \sum_{j+0}^{n-1} \left\{W\left(t_j^*\right) W\left(t_{j+1}\right) - W\left(t_j^*\right) W\left(t_j\right) \pm W\left(t_j^*\right)^2 \pm 2W\left(t_j\right) W\left(t_j^*\right) \pm W\left(t_j\right)^2\right\} \\
&= \lim_{||\Pi|| \to 0} \sum_{j+0}^{n-1} \left\{W\left(t_j^*\right) W\left(t_{j+1}\right) - W\left(t_j^*\right)^2\right\} + \lim_{||\Pi|| \to 0} \sum_{j+0}^{n-1} \left\{W\left(t_j\right) W\left(t_j^*\right) - W\left(t_j\right)^2\right\} \\
&\quad + \lim_{||\Pi|| \to 0} \sum_{j+0}^{n-1} \left\{W\left(t_j\right)^2 - 2W\left(t_j\right) W\left(t_j^*\right) + W\left(t_j^*\right)^2\right\} \\
&= \int_0^T W(t) dW(t) + \lim_{||\Pi|| \to 0} \sum_{j+0}^{n-1} \left(W\left(t_j\right)^2 - W\left(t_j^*\right)^2\right)^2 \\
&= \frac{1}{2} W^2(T) \qquad \text{(q.e.d.)}
\end{aligned}$$

In the last step, we have used the result that $\int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$, as e.g. shown in Example 4.3.2 and the quadratic variation of the Stratonovich integral that we calculated in part (i).

Exercise 4.5 (Solving the generalized geometric Brownian motion equation)

Let $f(t,x) = \ln x$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \frac{1}{x}, \quad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{x^2}$$

and

$$(dS(t))^2 = \sigma^2(t)S^2(t)dt.$$

Applying Itô's lemma, the differential of the log stock price $d\ln S(t)$ becomes

$$\begin{aligned} d\ln S(t) &= df(t,S(t)) \\ &= \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}(dS(t))^2 \\ &= \alpha(t)dt + \sigma(t)dW(t) - \frac{1}{2}\sigma^2(t)dt \\ &= \left(\alpha(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW(t). \end{aligned}$$

In integral form, we get

$$\ln S(t) = \ln S(0) + \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW(s). \tag{1}$$

Taking the exponential yields

$$S(t) = S(0)\exp\left\{\int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)dW(s)\right\}.$$

The distribution of $S(t)$ will in general not be log-normal. In order to have log-normality for $S(t)$, we require $\ln S(t)$ to be normally distributed. By Theorem 4.4.9, the Itô integral in Equation (1) is only guaranteed to be normally distributed if $\alpha(t)$ and $\sigma(t)$ are deterministic functions of time. In this case

$$\ln S(t) \sim \mathcal{N}\left(\int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s)\right)ds, \int_0^t \sigma^2(s)ds\right).$$

Note that this includes the less general case of a Geometric Brownian Motion with $\alpha(t) = \bar{\alpha}$ and $\sigma(t) = \bar{\sigma}$, i.e the drift and diffusion coefficient being a constant as it is assumed in the Black-Scholes model.

Exercise 4.6

Let $f(t,x) = S(0)e^x$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = f(x), \quad \frac{\partial^2 f}{\partial x^2} = f(x).$$

Now define

$$X(t) = \left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma W(t).$$

By the Itô formula, the differential of $S(t)$ is given by

$$\begin{aligned}
dS(t) &= df(t, X(t)) \\
&= S(t)dX(t) + \frac{1}{2}S(t)dX(t)dX(t) \\
&= \alpha S(t)dt - \frac{1}{2}\sigma^2 S(t)dt + \sigma S(t)dW(t) + \frac{1}{2}\sigma^2 S(t)dt \\
&= \alpha S(t)dt + \sigma S(t)dW(t).
\end{aligned}$$

Now let $f(t,x) = x^p$ such that

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = px^{p-1}, \quad \frac{\partial^2 f}{\partial x^2} = p(p-1)x^{p-1}.$$

Using the Itô formula to compute the differential of $d\left(S^p(t)\right)$ yields

$$\begin{aligned}
d\left(S^p(t)\right) &= d\left(f(t, S(t))\right) \\
&= pS^{p-1}(t)dS(t) + \frac{1}{2}p(p-1)S^{p-2}(t)dS(t)dS(t) \\
&= \alpha pS^p(t)dt + \sigma pS^p(t)dW(t) + \frac{1}{2}\sigma^2 p(p-1)S^p(t)dt \\
&= \left(\alpha + \frac{1}{2}(p-1)\right) pS^p(t)dt + \sigma pS^p(t)dW(t).
\end{aligned}$$

Exercise 4.7

(i) Let $f(t,x) = x^4$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 4x^3, \quad \frac{\partial^2 f}{\partial x^2} = 12x^2.$$

Using the Itô formula to compute the differential of $d\left(W^4(t)\right)$ yields

$$\begin{aligned} d\left(W^4(t)\right) &= 4W^3(t)dW(t) + 6W^2(t)dt \\ W^4(t) &= 4\int_0^T W^3(t)dW(t) + 6\int_0^T W^2(t)dt \end{aligned}$$

Note that there is in general a constant of integration in the second equation. But since $W^4(0) = 0$, we omitted it directly.

(ii) Taking expectations on both sides and using that an Itô integral is a martingale that starts at zero yields

$$\begin{aligned} \mathbb{E}\left[W^4(t)\right] &= 4\mathbb{E}\left[\int_0^T W^3(t)dW(t)\right] + 6\mathbb{E}\left[\int_0^T W^2(t)dt\right] \\ &= 6\int_0^T \mathbb{E}\left[W^2(t)\right]dt = 6\int_0^T tdt \\ &= 6\left|\frac{1}{2}t^2\right|_0^T = 3T^2 \qquad \text{(q.e.d.)} \end{aligned}$$

(iii) Let $f(t,x) = x^6$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 6x^5, \quad \frac{\partial^2 f}{\partial x^2} = 30x^4.$$

Using the Itô formula to compute the differential of $d\left(W^6(t)\right)$ yields

$$\begin{aligned} d\left(W^6(t)\right) &= 6W^5(t)dW(t) + 15W^4(t)dt \\ W^6(t) &= 6\int_0^T W^5(t)dW(t) + 15\int_0^T W^4(t)dt \end{aligned}$$

Taking expectations on both sides and using the result in part (ii) yields

$$\begin{aligned} \mathbb{E}\left[W^6(t)\right] &= 6\mathbb{E}\left[\int_0^T W^5(t)dW(t)\right] + 30\mathbb{E}\left[\int_0^T W^4(t)dt\right] \\ &= 15\int_0^T \mathbb{E}\left[W^4(t)\right]dt = 15\int_0^T 3t^2dt \\ &= 15\left|t^3\right|_0^T = 15T^3 \end{aligned}$$

Exercise 4.8 (Solving the Vasicek equation)

(i) Let $f(t,x) = e^{\beta t}x$. We have

$$\frac{\partial f}{\partial t} = \beta f(t,x), \quad \frac{\partial f}{\partial x} = e^{\beta t}, \quad \frac{\partial^2 f}{\partial x^2} = 0.$$

Using the Itô formula, we can compute the differential of $d\left(e^{\beta t}R(t)\right)$ to be

$$\begin{aligned} d\left(e^{\beta t}R(t)\right) &= df(t, R(t)) \\ &= \beta e^{\beta t}R(t)dt + e^{\beta t}dR(t) \\ &= \beta e^{\beta t}R(t)dt + e^{\beta t}\alpha dt - \beta e^{\beta t}R(t)dt + e^{\beta t}\sigma dW(t) \\ &= e^{\beta t}\alpha dt + e^{\beta t}\sigma dW(t) \end{aligned} \tag{2}$$

(ii) Integrating Equation (2) yields

$$\begin{aligned} e^{\beta t}R(t) &= R(0) + \int_0^t e^{\beta u}\alpha du + \int_0^t e^{\beta u}\sigma dW(u) \\ &= R(0) + \alpha \left| \frac{1}{\beta}e^{\beta u} \right|_0^t + \sigma \int_0^t e^{bu}dW(u) \\ &= R(0) + \frac{\alpha}{\beta}\left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{bu}dW(u) \end{aligned}$$

Thus, $R(t)$ is given by

$$R(t) = e^{-\beta t}R(0) + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) + e^{-\beta t}\sigma \int_0^t e^{bu}dW(u)$$

This is the same expression as given in Example 4.4.10.

Note that the solution strategy employed is very common for stochastic differential equations with a geometric drift term. Let $X_(t) = \alpha(t)X(t)dt + \ldots$. Then by computing the differential of $e^{-\alpha(t)t}X_(t)$, we can remove the geometric drift.

Exercise 4.9

(i)

$$
\begin{aligned}
Ke^{-r\tau}\mathcal{N}'(d_-) &= Ke^{-r\tau}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{d_-^2}{2}\right\} \\
&= K\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2\tau}\left(\ln\left(\frac{x}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)\tau\right)^2-r\tau\right\} \\
&= K\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2\tau}\left(\ln^2\left(\frac{x}{K}\right)+2\ln\left(\frac{x}{K}\right)\left(r-\frac{1}{2}\sigma^2\right)\tau\right.\right. \\
&\quad \left.\left.+\left(r-\frac{1}{2}\sigma^2\right)^2\tau^2+2r\sigma^2\tau^2\right)\right\} \\
&= K\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2\tau}\left(\ln^2\left(\frac{x}{K}\right)+2\ln\left(\frac{x}{K}\right)\left(r-\frac{1}{2}\sigma^2\right)\tau\right.\right. \\
&\quad \left.\left.+\left(r+\frac{1}{2}\sigma^2\right)^2\tau^2\right)\pm\ln\left(\frac{x}{K}\right)\right\} \\
&= K\frac{1}{\sqrt{2\pi}}\exp\left\{\ln\left(\frac{x}{K}\right)\right\}\exp\left\{-\frac{1}{2\sigma^2\tau}\left(\ln^2\left(\frac{x}{K}\right)\right.\right. \\
&\quad \left.\left.+2\ln\left(\frac{x}{K}\right)\left(r+\frac{1}{2}\sigma^2\right)\tau+\left(r+\frac{1}{2}\sigma^2\right)^2\tau^2\right)\right\} \\
&= x\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2\tau}\left(\ln\left(\frac{x}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)\tau\right)^2\right\} \\
&= x\mathcal{N}'(d_+) \qquad \text{(q.e.d.)}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
c_x &= \frac{\partial}{\partial x}\left[x\mathcal{N}(d_+)-Ke^{-r\tau}\mathcal{N}(d_-)\right] \\
&= \mathcal{N}(d_+)+x\mathcal{N}'(d_+)\frac{\partial d_+}{\partial x}-Ke^{-r\tau}\mathcal{N}'(d_-)\frac{\partial d_-}{\partial x} \\
&= \mathcal{N}(d_+) \qquad \text{(q.e.d.)}
\end{aligned}
$$

Here, we used that

$$\frac{\partial d_+}{\partial x}=\frac{\partial d_-}{\partial x}=\frac{1}{x\sigma\sqrt{\tau}}$$

as well as the result from (i).

(iii) First, note that

$$\begin{aligned}
\frac{\partial d_\pm}{\partial t} &= \frac{\partial}{\partial t}\frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{x}{K}\right)+\left(r\pm\frac{1}{2}\sigma^2\right)(T-t)\right] \\
&= \frac{1}{2\sigma\sqrt{(T-t)^3}}\left[\ln\left(\frac{x}{K}\right)+\left(r\pm\frac{1}{2}\sigma^2\right)(T-t)\right]-\frac{1}{\sigma\sqrt{T-t}}\left(r\pm\frac{1}{2}\sigma^2\right)
\end{aligned}$$

and consequently

$$\frac{\partial d_+}{\partial t}-\frac{\partial d_-}{\partial t} = -\frac{\sigma}{2\sqrt{T-t}}.$$

We thus get

$$\begin{aligned}
c_t &= \frac{\partial}{\partial t}\left[x\mathcal{N}(d_+)-Ke^{-r(T-t)}\mathcal{N}(d_-)\right] \\
&= x\mathcal{N}'(d_+)\frac{\partial d_+}{\partial t}-rKe^{-r(T-t)}\mathcal{N}(d_-)-Ke^{-r(T-t)}\mathcal{N}'(d_-)\frac{\partial d_-}{\partial t} \\
&= x\mathcal{N}'(d_+)\left(\frac{\partial d_+}{\partial t}-\frac{\partial d_-}{\partial t}\right)-rKe^{-r(T-t)}\mathcal{N}(d_-) \\
&= -\frac{\sigma x}{2\sqrt{T-t}}\mathcal{N}'(d_+)-rKe^{-r(T-t)}\mathcal{N}(d_-) \qquad \text{(q.e.d.).}
\end{aligned}$$

(iv) We first need to compute $c_x x$, i.e. the gamma of the option.

$$c_{xx}=\mathcal{N}'(d_+)\frac{\partial d_+}{\partial x} = \frac{1}{x\sigma\sqrt{\tau}}\mathcal{N}'(d_+)$$

Substituting into the PDE yields

$$\begin{aligned}
& c_t(t,x)+rxc_x(t,x)+\frac{1}{2}\sigma^2c_{xx}(t,x)-rc(t,x) \\
=\ & -\frac{\sigma x}{2\sqrt{\tau}}\mathcal{N}'(d_+)-rKe^{-r\tau}\mathcal{N}(d_-)+rx\mathcal{N}(d_+)+\frac{\sigma x}{2\sqrt{\tau}}\mathcal{N}'(d_+)-rx\mathcal{N}(d_+) \\
& +rKe^{-r\tau}\mathcal{N}(d_-) \\
=\ & 0 \qquad \text{(q.e.d.).}
\end{aligned}$$

(v)

(vi)

(vii)

Exercise 4.11

When dynamically replicating an mispriced option in the Black-Scholes model, we have to decide upon which volatility to use to compute the hedge ratio. In this case, the market uses a volatility of σ_1 to price the option at all times $t \in [0, T]$. Knowing that all model assumption hold but with the actual volatility of the underlying diffusion being $\sigma_2 > \sigma_1$, we will enter a long position in the underpriced european call option and dynamically delta hedge it by shorting $\Delta(t) = \frac{\partial c}{\partial S}$ shares of the underlying. But since the delta itself depends upon the volatility, there are two possibilities to do this: Either we use the market implied volatility (σ_1) or the actual volatility (σ_2) to compute the hedge ratio. These two approaches will in general lead to different returns. As implicitly suggested by the exercise, we will use the delta calculated based upon the implied volatility to hedge the long position in the european call option.

Let $f(t, x) = e^{-rt}x$. We have

$$\frac{\partial f}{\partial t} = -rf(t, x), \quad \frac{\partial f}{\partial x} = e^{-rt}, \quad \frac{\partial^2 f}{\partial x^2} = 0.$$

Apply Itô's lemma to the differential of the discounted portfolio value $d\left(e^{-rt}X(t)\right)$

$$\begin{aligned} d\left(e^{-rt}X(t)\right) &= df(t, X(t)) \\ &= -re^{-rt}X(t)dt + e^{-rt}dX(t) \\ &= e^{-rt}\left(-rX(t)dt + dX(t)\right). \end{aligned} \tag{3}$$

We furthermore note that the differential of the call option price $dc(t, S(t))$ evolves according to

$$\begin{aligned} dc(t, S(t)) &= \frac{\partial c}{\partial t}dt + \frac{\partial c}{\partial S}dS(t) + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\left(dS(t)\right)^2 \\ &= \left(\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_2^2 S^2(t)\frac{\partial^2 c}{\partial S^2}\right)dt + \frac{\partial c}{\partial S}dS(t). \end{aligned} \tag{4}$$

Plugging the given differential of the portfolio value and the differential of the call option price in Equation (4) back into Equation (3) yields

$$
\begin{aligned}
d\left(e^{-rt}X(t)\right) &= e^{-rt}\Bigg[-rX(t)dt + \left(\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_2^2 S^2(t)\frac{\partial^2 c}{\partial S^2}\right)dt + \frac{\partial c}{\partial S}dS(t) \\
&\quad -\frac{\partial c}{\partial S}dS(t) + rX(t)dt - rc(t,S(t))dt + rS(t)\frac{\partial c}{\partial S}dt \\
&\quad -\frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)S^2(t)\frac{\partial^2 c}{\partial S^2}dt\Bigg] \\
&= e^{-rt}\left[\frac{\partial c}{\partial t} + \frac{1}{2}\sigma_1^2 S^2(t)\frac{\partial^2 c}{\partial S^2} + rS(t)\frac{\partial c}{\partial S} - rc(t,S(t))\right]dt.
\end{aligned}
$$

We note that the call price $c(t, S(t))$ is a specific solution to the Black-Scholes PDE where the volatility is equal to σ_1

$$
\frac{\partial c}{\partial t} + rS(t)\frac{\partial c}{\partial S} + \frac{1}{2}\sigma_1^2 S^2(t)\frac{\partial^2 c}{\partial S^2} = rc(t,S(t))
$$

for the payoff-specific terminal condition $c(T, S(T)) = \max(S(T)-K, 0), S(T) \geq 0$ and the boundary conditions $c(t, 0) = 0, 0 \leq t \leq T$ and $\lim_{S(t)\to\infty}\left\{c(t,S(t)) - \left(S(t) - Ke^{-r(T-t)}\right)\right\} = 0, 0 \leq t \leq T$. Using this result, it directly follows that $d\left(e^{-rt}X(t)\right) = 0$.

Exercise 4.13 (Decomposition of correlated Brownian motions into independent Brownian motions)

The differential form of $B_1(t)$ and $B_2(t)$ are given by

$$
\begin{aligned}
dB_1(t) &= dW_1(t) \\
dB_2(t) &= \rho(t)dW_1(t) + \sqrt{1-\rho^2(t)}dW_2(t).
\end{aligned}
$$

This allows us to express the differentials of $W_1(t)$ and $W_2(t)$ in terms of the changes in the correlated Brownian motions $B_1(t)$ and $B_2(t)$

$$
\begin{aligned}
dW_1(t) &= dB_1(t) \\
dW_2(t) &= -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dB_1(t) + \frac{1}{\sqrt{1-\rho^2(t)}}dB_2(t).
\end{aligned} \tag{5}
$$

We have to show that these two processes are independent Brownian motions. While it is obvious that $W_1(t)$ is a Brownian motion since it is just equal to $B_1(t)$, we have to check if this also holds for $W_2(t)$. According to Lévy's theorem, any continuous martingale $M(t)$ with $M(0) = 0$ and quadratic variation equal to $[M, M](t) = t$ (i.e. $dM(t)dM(t) = dt$) is a

Brownian motion. Furthermore, according to the extended Lévy theorem, two Brownian motions, with zero cross variation (i.e. $dW_1(t)dW_2(t) = 0$) are independent.

(i) **Continuity:**

The continuity of $W_2(t)$ follows from the continuity of the two Brownian motions $B_1(t)$ and $B_2(t)$. To see this explicitly, we write down the integral form of Equation (5) to get

$$W_2(t) = W_2(0) - \int_0^t \frac{\rho(s)}{\sqrt{1-\rho^2(s)}} dB_1(s) + \int_0^t \frac{1}{\sqrt{1-\rho^2(s)}} dB_2(s). \qquad (6)$$

Since an Itô integral is a continuous function of its upper limit of integration, $W_2(t)$ is continuous as well.

(ii) **Martingale property:**

The martingale property of $W_2(t)$ directly follows from the absence of a drift term in Equation (5) or its representation as an Itô integral in Equation (6).

(iii) **Starting at zero:**

Given the information in the exercise, we can't show that $W_2(0) = 0$. When integrating Equation (5), we have to add a constant of integration that might take any value in $\mathbb{R}$. Similarly, we could argue that in the equation for $dB_2(t)$, only the differential $dW_2(t)$ and thus its initial value does not matter.

(iv) **Unit quadratic variation:**

The quadratic variation of $W_2(t)$ can be computed as

$$\begin{aligned} dW_2(t)dW_2(t) &= \frac{\rho^2(t)}{1-\rho^2(t)}dt + \frac{1}{1-\rho^2(t)}dt - 2\frac{\rho^2(t)}{1-\rho^2(t)}dt \\ &= \frac{1-\rho^2(t)}{1-\rho^2(t)}dt = dt \quad \text{(q.e.d.)}. \end{aligned}$$

and it follows that $[W_2, W_2](t) = t$.

(v) **Zero cross variation:**

Using $dB_1(t)dB_1(t) = dt$ and $dB_1(t)dB_2(t) = \rho(t)dt$, the instantaneous cross variation between $W_1(t)$ and $W_2(t)$ can be computed as

$$dW_1(t)dW_2(t) = -\frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dt + \frac{\rho(t)}{\sqrt{1-\rho^2(t)}}dt = 0.$$

This establishes that $W_1(t)$ and $W_2(t)$ have zero cross-covariance. An alternative proof of this result directly computes $dB_1(t)dB_2(t)$

$$dB_1(t)dB_2(t) \quad = \quad \rho(t)dt + \sqrt{1-\rho^2(t)}dW_1(t)dW_2(t).$$

This equation is only equal to $\rho(t)dt$ as given in the exercise, when

$$\sqrt{1-\rho^2(t)}dW_1(t)dW_2(t) = 0.$$

Since $\rho \in (-1, 1)$, the square root term can never be equal to zero and we conclude that $dW_1(t)dW_2(t) = 0$.

Exercise 4.15 (Creating correlated Brownian motions from independent ones)

(i) Similar to Exercise 4.13, we use Lévy's theorem to show that $B_i(t)$ is a Brownian motion for each $i = 1, \dots, m$. Since $W_j(0) = 0$ for each $j = 1, \dots, d$, it directly follows from the definition of $B_i(t)$ that $B_i(0) = 0$. Furthermore, due to its representation as a sum of Itô integrals, we obtain both the continuity and the martingale property. It remains to show that $B_i(t)$ has quadratic variation $[B_i, B_i](t) = t$ or equivalently that $dB_i(t)dB_i(t) = dt$. In differential form, $B_i(t)$ is

$$dB_i(t) = \frac{1}{\sigma_i(t)} \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t).$$

Using the independence of the d-dimensional vector of Brownian motions, we get

$$dB_i(t)dB_i(t) = \frac{1}{\sigma_i^2(t)} \sum_{j=1}^{d} \sigma_{ij}^2(t)dt = \frac{\sum_{j=1}^{d} \sigma_{ij}^2(t)}{\sum_{j=1}^{d} \sigma_{ij}^2(t)} dt = dt \quad \text{(q.e.d.)}.$$

(ii) Again using independence we get

$$dB_i(t)dB_k(t) = \frac{1}{\sigma_i(t)\sigma_k(t)} \sum_{j=1}^{d} \sigma_{ij}\sigma_{kj}(t)dt = \rho_{ik}(t)dt.$$

Exercise 4.16 (Creating independent Brownian motions to represent correlated ones)

In differential form and using matrix notation, we have $dB(t) = A(t)dW(t)$ or $dW(t) = A^{-1}(t)dB(t)$. We use the multidimensional Lévy theorem (a straight forward extension of Theorem 4.6.5) to show that $W(t)$ is a vector of independent Brownian motions. We need to check the following conditions:

- **Martingale property:**
  Since we can represent $W(t)$ in terms of an integral constant plus and Itô integral

  $$W(t) = W(0) + \int_0^t A^{-1}(u)dB(u) \qquad \Leftrightarrow \qquad W_i(t) = W_i(0) + \sum_{j=1}^{m} \int_0^t a_{ij}^{-1}(u)dB_j(u)$$

  it follows that each element $W_i(t)$ of this vector is a martingale (Theorem 4.3.1(iv)).

- **Continuity:**
  By Theorem 4.3.1(i), Itô integrals have continuous sample paths.

- **Starting at zero:**
  Similar to Exercise 4.13(iii), we can't show that $W(0)$ is a vector of zeros. In the first point, we could choose the integration constant $W(0)$ arbitrarily as only the change $dW(t)$ is relevant in the definition of $B(t)$. However, the exercise is only asking for the existence of a vector of Brownian motions $W(t)$. Thus, by setting $W(0) = 0$ we can find such a vector.

- **Unit quadratic variation and zero cross variation:**

We need to show that each $W_i(t)$ has unit quadratic variation but zero cross variation with any other $W_j(t)$ for $j \neq i$ or equivalently that $dW(t)dW^T(t) = I$ where I is the $m \times m$ identity matrix. This can be easily done using matrix notation. We get

$$\begin{aligned}
dW(t)dW^T(t) &= A^{-1}(t)dB(t)\left(A^{-1}(t)dB(t)\right)^T \\
&= A^{-1}(t)dB(t)dB^T(t)A^{-T}(t) \\
&= A^{-1}(t)C(t)A^{-T}(t) \\
&= A^{-1}(t)A(t)A^T(t)A^{-T} \\
&= I \qquad \text{(q.e.d.)}.
\end{aligned}$$

Exercise 4.17 (Instantaneous correlation)

(i) We first note that

$$B_1(t_0+\epsilon)\,B_2(t_0+\epsilon) = B_1(t_0)\,B_2(t_0) + \int_{t_0}^{t_0+\epsilon} d\left(B_1(s)B_2(s)\right).$$

By Corollary 4.6.3, we have for the differential of the product of the two Brownian motions

$$\begin{aligned}
d\left(B_1(t)B_2(t)\right) &= B_2(t)dB_1(t) + B_1(t)dB_2(t) + dB_1(t)dB_2(t) \\
&= B_2(t)dB_1(t) + B_1(t)dB_2(t) + \rho dt.
\end{aligned}$$

Thus,

$$B_1(t_0+\epsilon)\,B_2(t_0+\epsilon) = B_1(t_0)\,B_2(t_0) + \int_{t_0}^{t_0+\epsilon} B_2(s)dB_1(s) + \int_{t_0}^{t_0+\epsilon} B_1(s)dB_2(s) + \rho\epsilon$$

and

$$\mathbb{E}\left[B_1(t_0+\epsilon)B_2(t_0+\epsilon)\middle|\mathcal{F}(t_0)\right] = B_1(t_0)B_2(t_0)+\rho\epsilon$$

where we used that the two integrals are independent of the filtration $\mathcal{F}(t_0)$ and that by Theorem 4.3.1(iv) Itô integrals are martingales. Finally,

$$\begin{aligned}
& \mathbb{E}\left[\left(B_1(t_0+\epsilon)-B_1(t_0)\right)\left(B_2(t_0+\epsilon)-B_2(t_0)\right)\middle|\mathcal{F}(t_0)\right] \\
= \; & \mathbb{E}\left[B_1(t_0+\epsilon)B_2(t_0+\epsilon)\middle|\mathcal{F}(t_0)\right]-\mathbb{E}\left[B_1(t_0+\epsilon)B_2(t_0)\middle|\mathcal{F}(t_0)\right] \\
& -\mathbb{E}\left[B_1(t_0)B_2(t_0+\epsilon)\middle|\mathcal{F}(t_0)\right]+\mathbb{E}\left[B_1(t_0)B_2(t_0)\middle|\mathcal{F}(t_0)\right] \\
= \; & B_1(t_0)B_2(t_0)+\rho\epsilon-B_2(t_0)\mathbb{E}\left[B_1(t_0+\epsilon)\right] \\
& -B_1(t_0)\mathbb{E}\left[B_2(t_0+\epsilon)\right]+B_1(t_0)B_2(t_0) \\
= \; & \rho\epsilon. \qquad \text{(q.e.d.)}
\end{aligned}$$

(ii) The means are

$$\begin{aligned}
M_i(\epsilon) &= \mathbb{E}\left[X_i(t_0+\epsilon)-X_i(t_0)\middle|\mathcal{F}(t_0)\right] \\
&= \int_{t_0}^{t_0+\epsilon}\Theta_i du \\
&= \Theta_i\epsilon \qquad \text{(q.e.d.)}.
\end{aligned}$$

Where we again used that the Riemann integral is deterministic and that the Itô integral is a martingale with zero expected value.

The variances are

$$\begin{aligned}
V_i(\epsilon) &= \mathbb{E}\left[\left(X_i(t_0+\epsilon)-X_i(t_0)\right)^2\middle|\mathcal{F}(t_0)\right]-M_i^2(\epsilon) \\
&= \mathbb{E}\left[\left(\int_{t_0}^{t_0+\epsilon}\Theta_i du+\int_{t_0}^{t_0+\epsilon}\sigma_i dB_i(u)\right)^2\right]-\Theta_i^2\epsilon^2 \\
&= \left(\int_{t_0}^{t_0+\epsilon}\Theta_i du\right)\mathbb{E}\left[\int_{t_0}^{t_0+\epsilon}\sigma_i dB_i(u)\right]+\mathbb{E}\left[\left(\int_{t_0}^{t_0+\epsilon}\sigma_i dB_i(u)\right)^2\right] \\
&= \int_{t_0}^{t_0+\epsilon}\sigma_i^2 du \\
&= \sigma_i^2\epsilon \qquad \text{(q.e.d.)}.
\end{aligned}$$

Here, we used that the increment in the Itô processes is independent of the filtration $\mathcal{F}(t_0)$ in the second step. We can take the deterministic time integral out of the expectation in the third step and again reply on the martingale property of the Itô integral in the fourth step to eliminate this term. We further use the Itô isometry (Theorem 4.3.1(v)) to transform the squared Itô integral into a Riemann integral in the fourth step.

The covariance is

$$
\begin{aligned}
C(\epsilon) &= \mathbb{E}\left[\left(X_1(t_0+\epsilon)-X_1(t_0)\right)\left(X_2(t_0+\epsilon)-X_2(t_0)\right)\middle|\,\mathcal{F}(t_0)\right]-M_1(\epsilon)M_2(\epsilon)\\
&= \mathbb{E}\left[\left(\int_{t_0}^{t_0+\epsilon}\Theta_1 du+\int_{t_0}^{t_0+\epsilon}\sigma_1 dB_1(u)\right)\left(\int_{t_0}^{t_0+\epsilon}\Theta_2 du+\int_{t_0}^{t_0+\epsilon}\sigma_2 dB_2(u)\right)\right]-\Theta_1\Theta_2\epsilon^2\\
&= \left(\int_{t_0}^{t_0+\epsilon}\Theta_1 du\right)\mathbb{E}\left[\int_{t_0}^{t_0+\epsilon}\sigma_2 dB_2(u)\right]+\left(\int_{t_0}^{t_0+\epsilon}\Theta_2 du\right)\mathbb{E}\left[\int_{t_0}^{t_0+\epsilon}\sigma_1 dB_1(u)\right]\\
&\quad +\mathbb{E}\left[\left(\int_{t_0}^{t_0+\epsilon}\sigma_1 dB_1(u)\right)\left(\int_{t_0}^{t_0+\epsilon}\sigma_2 dB_2(u)\right)\right]\\
&= \mathbb{E}\left[\left(\int_{t_0}^{t_0+\epsilon}\sigma_1 dB_1(u)\right)\left(\int_{t_0}^{t_0+\epsilon}\sigma_2 dB_2(u)\right)\right]
\end{aligned}
$$

In order to solve the remaining expectation, we first define for $t \geq t_0$

$$
I_i(t)=\int_{t_0}^{t}\sigma_i dB_i(u) \quad \Rightarrow \quad dI_i(t)=\sigma_i dB_i(t)
$$

By the Itô product rule, we have

$$
d\left(I_1(t)I_2(t)\right)(t)=I_2(t)dI_1(t)+I_1(t)dI_2(t)+dI_1(t)dI_2(t)
$$

or

$$
\begin{aligned}
I_1(t)I_2(t) &= I_1(t_0)I_2(t_0)+\int_{t_0}^{t}d\left(I_1(u)I_2(u)\right)\\
&= \int_{t_0}^{t}I_2(u)dI_1(u)+\int_{t_0}^{t}I_1(u)dI_2(u)+\int_{t_0}^{t}\rho\sigma_1\sigma_2 du.
\end{aligned}
$$

Thus,

$$C(\epsilon) = \mathbb{E}\left[I_1\left(t_0+\epsilon\right) I_2\left(t_0+\epsilon\right)\right] = \int_{t_0}^{t_0+\epsilon} \rho\sigma_1\sigma_2 du = \rho\sigma_1\sigma_2\epsilon. \qquad \text{(q.e.d.)}.$$

Consequently, we have for the correlation between the increments of the Itô processes

$$\frac{C(\epsilon)}{\sqrt{V_1(\epsilon)V_2(\epsilon)}} = \frac{\rho\sigma_1\sigma_2\epsilon}{\sqrt{\sigma_1^2\sigma_2^2\epsilon_2}} = \rho.$$

(iii) We have

$$\begin{aligned} M_i(\epsilon) &= \mathbb{E}\left[X_i\left(t_0+\epsilon\right) - X\left(t_0\right) \middle| \mathcal{F}\left(t_0\right)\right] \\ &= \mathbb{E}\left[\int_{t_0}^{t_0+\epsilon} \Theta_i(u)du + \int_{t_0}^{t_0+\epsilon} \sigma_i(u)dB_i(u) \middle| \mathcal{F}\left(t_0\right)\right] \\ &= \mathbb{E}\left[\int_{t_0}^{t_0+\epsilon} \Theta_i(u)du \middle| \mathcal{F}\left(t_0\right)\right]. \end{aligned}$$

We first note that by Definition 1.2.1, the real constant M can be considered a (trivial) random variable as well. Here $\{M \in B\} = \Omega$, i.e. we do not learn anything about the particular outcome $\omega \in \Omega$ by observing M. We now apply the dominated convergence theorem (Theorem 1.4.9) to the sequence of random variables

$$I(\epsilon) = \frac{1}{\epsilon}\int_{t_0}^{t_0+\epsilon} \Theta_i(u)du$$

that is bounded in absolute value by M. We have

$$\begin{aligned} \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon} M_i(\epsilon) &= \lim_{\epsilon\downarrow 0} \mathbb{E}\left[I(\epsilon) \middle| \mathcal{F}\left(t_0\right)\right] \\ &= \mathbb{E}\left[\lim_{\epsilon\downarrow 0} I(\epsilon) \middle| \mathcal{F}\left(t_0\right)\right] \\ &= \mathbb{E}\left[\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left[I\left(0\right) + I'\left(0\right)\epsilon + o\left(\epsilon\right)\right] \middle| \mathcal{F}\left(t_0\right)\right] \\ &= \mathbb{E}\left[\Theta_i\left(t_0\right) \middle| \mathcal{F}\left(t_0\right)\right] \\ &= \Theta_i\left(t_0\right) \qquad \text{(q.e.d.)}. \end{aligned}$$

Here, we used a Taylor series expansion of the integral around $\epsilon = 0$ where $I(0) = 0$ and $I'(0) = \Theta_i\left(t_0\right)$.

(iv) Similar to the hint, we define for $t > t_0$

$$Y_i(t) = \int_{t_0}^{t} \sigma_i(u) dB_i(u)$$

and get

$$\begin{aligned}
D_{ij}(\epsilon) &= \mathbb{E}\left[\left(X_i\left(t_0+\epsilon\right)-X_i\left(t_0\right)\right)\left(X_j\left(t_0+\epsilon\right)-X_j\left(t_0\right)\right) \middle| \mathcal{F}\left(t_0\right)\right] - M_i(\epsilon) M_j(\epsilon) \\
&= \mathbb{E}\left[\left(Y_i\left(t_0+\epsilon\right)+\int_{t_0}^{t_0+\epsilon} \Theta_i(u) du\right)\left(Y_j\left(t_0+\epsilon\right)+\int_{t_0}^{t_0+\epsilon} \Theta_j(u) du\right) \middle| \mathcal{F}\left(t_0\right)\right] \\
&\quad -M_i(\epsilon) M_j(\epsilon) \\
&= \mathbb{E}\left[Y_i\left(t_0+\epsilon\right) Y_j\left(t_0+\epsilon\right) \middle| \mathcal{F}\left(t_0\right)\right]+\left(\int_{t_0}^{t_0+\epsilon} \Theta_j(u) du\right) \mathbb{E}\left[Y_i\left(t_0+\epsilon\right) \middle| \mathcal{F}\left(t_0\right)\right] \\
&\quad +\left(\int_{t_0}^{t_0+\epsilon} \Theta_i(u) du\right) \mathbb{E}\left[Y_j\left(t_0+\epsilon\right) \middle| \mathcal{F}\left(t_0\right)\right] \\
&= \mathbb{E}\left[Y_i\left(t_0+\epsilon\right) Y_j\left(t_0+\epsilon\right) \middle| \mathcal{F}\left(t_0\right)\right]
\end{aligned}$$

Here, we followed the same main steps as in the derivation in (ii). We now apply the Itô product rule again to compute the remaining expectation and get

$$\begin{aligned}
D_{ij}(\epsilon) &= \mathbb{E}\left[\int_{t_0}^{t_0+\epsilon} Y_j(u) dY_i(u)+\int_{t_0}^{t_0+\epsilon} Y_i(u) dY_j(u)+\int_{t_0}^{t_0+\epsilon} dY_i(u) dY_j(u) \middle| \mathcal{F}\left(t_0\right)\right] \\
&= \mathbb{E}\left[\int_{t_0}^{t_0+\epsilon} \rho_{ij}(u) \sigma_i(u) \sigma_j(u) du \middle| \mathcal{F}\left(t_0\right)\right].
\end{aligned}$$

We now apply the same limit argument as in (iii) and start by defining the sequence of random variables

$$I_{ij}(\epsilon) = \frac{1}{\epsilon} \int_{t_0}^{t_0+\epsilon} \rho_{ij}(u) \sigma_i(u) \sigma_j(u) du$$

that is bounded in absolute value by M^3. We get

$$
\begin{aligned}
\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}D_{ij}(\epsilon) &= \lim_{\epsilon\downarrow 0}\mathbb{E}\left[I_{ij}(\epsilon)\middle|\mathcal{F}(t_0)\right] \\
&= \mathbb{E}\left[\lim_{\epsilon\downarrow 0} I(\epsilon)\middle|\mathcal{F}(t_0)\right] \\
&= \mathbb{E}\left[\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\left[I(0)+I'(0)\epsilon+o(\epsilon)\right]\middle|\mathcal{F}(t_0)\right] \\
&= \mathbb{E}\left[\rho_{ij}(t_0)\,\sigma_i(t_0)\,\sigma_j(t_0)\middle|\mathcal{F}(t_0)\right] \\
&= \rho_{ij}(t_0)\,\sigma_i(t_0)\,\sigma_j(t_0) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

(v) We showed this already implicitly in the derivation of the means and covariances. We have

$$
\begin{aligned}
D_{ij}(\epsilon) &= \mathbb{E}\left[I_{ij}(\epsilon)\middle|\mathcal{F}(t_0)\right] \\
&= \mathbb{E}\left[I(0)+I'(0)\epsilon+o(\epsilon)\middle|\mathcal{F}(t_0)\right] \\
&= \mathbb{E}\left[\rho_{ij}(t_0)\,\sigma_i(t_0)\,\sigma_j(t_0)\,\epsilon+o(\epsilon)\middle|\mathcal{F}(t_0)\right] \\
&= \rho_{ij}(t_0)\,\sigma_i(t_0)\,\sigma_j(t_0)\,\epsilon+o(\epsilon) \\
&= \begin{cases} V_i(\epsilon)=\sigma_k^2(t_0)\,\epsilon+o(\epsilon) & \text{for } i=j=k \\ C(\epsilon)=\rho(t_0)\,\sigma_1(t_0)\,\sigma_2(t_0)\,\epsilon+o(\epsilon) & \text{for } i\neq j \end{cases} \qquad \text{(q.e.d.)}.
\end{aligned}
$$

(vi) Finally,

$$
\begin{aligned}
\lim_{\epsilon\downarrow 0}\frac{C(\epsilon)}{\sqrt{V_1(\epsilon)V_2(\epsilon)}} &= \lim_{\epsilon\downarrow 0}\frac{\rho(t_0)\,\sigma_1(t_0)\,\sigma_2(t_0)\,\epsilon+o(\epsilon)}{\sqrt{\left(\sigma_1^2(t_0)\,\epsilon+o(\epsilon)\right)\left(\sigma_2^2(t_0)\,\epsilon+o(\epsilon)\right)}} \\
&= \lim_{\epsilon\downarrow 0}\frac{\rho(t_0)\,\sigma_1(t_0)\,\sigma_2(t_0)}{\sqrt{\sigma_1^2(t_0)\,\sigma_2^2(t_0)}}+o(\epsilon) \\
&= \rho(t_0) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

Exercise 4.18

(i) Let $f(t,x)=\exp\left\{-\theta x-\left(r+\frac{1}{2}\theta^2\right)t\right\}$. We have

$$
\frac{\partial f}{\partial t}=-\left(r+\frac{1}{2}\theta^2\right)f(t,x),\quad \frac{\partial f}{\partial x}=-\theta f(t,x),\quad \frac{\partial^2 f}{\partial x^2}=\theta^2 f(t,x).
$$

Itô's lemma yields

$$\begin{aligned}
d\zeta(t) &= df(t, W(t)) \\
&= -\left(r + \frac{1}{2}\theta^2\right)\zeta(t)dt - \theta\zeta(t)dW(t) + \frac{1}{2}\theta^2\zeta(t)dt \\
&= -r\zeta(t)dt - \theta\zeta(t)dW(t) \quad \text{(q.e.d.)}.
\end{aligned}$$

(ii) By Itô's product rule (Corollary 4.6.3) we get

$$\begin{aligned}
d\left(\zeta(t)X(t)\right) &= \zeta(t)dX(t) + X(t)d\zeta(t) + dX(t)d\zeta(t) \\
&= r\zeta(t)X(t)dt + \zeta(t)\Delta(t)(\alpha - r)S(t)dt + \zeta(t)\Delta(t)\sigma S(t)dW(t) \\
&\quad -\theta\zeta(t)X(t)dW(t) - r\zeta(t)X(t)dt - \theta\zeta(t)\Delta(t)\sigma S(t)dt \\
&= \zeta(t)\left(\Delta(t)\sigma S(t) - \theta X(t)\right)dW(t). \qquad (7)
\end{aligned}$$

In the last step, we have used the fact that $\theta\sigma = \alpha - r$. Equation (7) shows that $\zeta(t)X(t)$ is a martingale, since all dt-terms vanish in the differential. Equivalently, we could express $\zeta(t)X(t)$ by a single Itô integral. Since all Itô integrals are martingales, so is $\zeta(t)X(t)$

$$\zeta(t)X(t) = \zeta(0)X(0) + \int_0^t \zeta(s)\left(\Delta(s)\sigma S(s) - \theta X(s)\right)dW(s).$$

(iii) First, note that

$$\zeta(0) = \exp\left\{-\theta W(0)\right\} = 1.$$

Now let $\Delta(t)$ be an adapted portfolio process such that, $X(T) = V(T)$, i.e. the final value of the trading strategy is equal to the final value of the $\mathcal{F}(T)$ measurable random variable $V(T)$. As has been shown in (ii), $\zeta(t)X(t)$ is a martingale, i.e.

$$\mathbb{E}\left[\zeta(t)X(t)|\mathcal{F}(s)\right] = \zeta(s)X(s), \quad s \leq t.$$

We obtain

$$X(0) = \zeta(0)X(0) = \mathbb{E}\left[\zeta(T)X(T)\right] = \mathbb{E}\left[\zeta(T)V(T)\right].$$

Exercise 4.19

(i) By Lévy's theorem it is sufficient to show that $B(t)$ is a continuous martingale starting at zero with unit quadratic variation in order to establish that it is a Brownian motion. We first note that $B(t)$ is an Itô integral with $B(0) = 0$. By Theorem 4.3.1, Itô integrals are continuous martingales. The quadratic variation of $B(t)$ is

$$(dB(t))^2 = (\text{sign}\,(W(t))\,dW(t))^2 = \text{sign}^2\,(W(t))\,(dW(t))^2 = dt$$

and it follows that it is a Brownian motion.

(ii) We have

$$\begin{aligned} d\,(B(t)W(t)) &= B(t)dW(t) + W(t)dB(t) + dB(t)dW(t) \\ &= B(t)dW(t) + W(t)\text{sign}\,(W(t))\,dW(t) + \text{sign}\,(W(t))\,dt \end{aligned}$$

and since $B(0)W(0) = 0$ we get

$$B(t)W(t) = \int_0^t B(s)dW(s) + \int_0^t W(s)\text{sign}\,(W(s))\,dW(s) + \int_0^t \text{sign}\,(W(s))\,ds$$

Since Itô integrals are martingales it follows that

$$\mathbb{E}\,[B(t)W(t)] = \mathbb{E}\left[\int_0^t \text{sign}\,(W(s))\,ds\right] = \int_0^t \mathbb{E}\,[\text{sign}\,(W(s))]\,ds = 0.$$

In the last step, we have used that with probability one half, $\text{sign}\,(W(t)) = \pm 1$ for $t > 0$.

(iii) Let $f(t,x) = x^2$. We have

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial^2 f}{\partial x^2} = 2.$$

Using the Itô formula to compute the differential of $dW^2(t)$ yields

$$dW^2(t) = 2W(t)dW(t) + dt \qquad \text{(q.e.d.)}$$

(iv) We have

$$
\begin{aligned}
d\left(B(t)W^2(t)\right) &= B(t)dW^2(t) + W^2(t)dB(t) + dB(t)dW^2(t) \\
&= 2B(t)W(t)dW(t) + B(t)dt + W^2(t)\text{sign}\left(W(t)\right)dW(t) \\
&\quad +2W(t)\text{sign}\left(W(t)\right)dt.
\end{aligned}
$$

Integrating both sides yields

$$
\begin{aligned}
B(t)W^2(t) &= \int_0^t 2B(s)W(s)dW(s) + \int_0^t B(s)ds + \int_0^t W^2(s)\text{sign}\left(W(s)\right)dW(s) \\
&\quad + \int_0^t 2W(s)\text{sign}\left(W(s)\right)ds.
\end{aligned}
$$

We take expectation on both sides and again use the martingale property of the Itô integral to get

$$
\mathbb{E}\left[B(t)W^2(t)\right] = \mathbb{E}\left[\int_0^t B(s)ds + \int_0^t 2W(s)\text{sign}\left(W(s)\right)ds\right] \tag{8}
$$

In order to evaluate the first Riemann integral, we set $f(t,x) = tx$ such that $f_t(t,x) = x$ and $f_x(t,x) = t$. We apply the Itô formula to obtain the differential of $tW(t)$ as

$$
d\left(tB(t)\right) = B(t)dt + tdB(t).
$$

Integrating and rearranging yields

$$
\int_0^T B(t)dt = TB(T) - \int_0^T tdB(t) = \int_0^T (T-t)dB(t).
$$

The last term is an Itô integral of a deterministic integrand and thus normally distributed with zero mean and variance

$$
\int_0^T (T-t)^2 dt = -\frac{1}{3}(T-t)^3\Big|_0^T = \frac{1}{3}T^3.
$$

It follows that the first term in Equation (8) is zero and we get

$$
\mathbb{E}\left[B(t)W^2(t)\right] = \mathbb{E}\left[\int_0^t 2W(s)\text{sign}\left(W(s)\right)ds\right] = \mathbb{E}\left[\int_0^t 2\left|W(s)\right|ds\right]
$$

It is obvious, that the expected value is strictly positive for $t > 0$ since the integrand is non-negative and almost surely strictly positive. If $B(t)$ and $W(t)$ are independent, then by Theorem 2.2.5 any two function $f(B(t))$ and $g(W(t))$ are independent. Now let $g(B(t)) = B(t)$ and $f(W(t)) = W^2(t)$ then by Theorem 2.2.7 independence implies that $\mathbb{E}[B(t)W^2(t)] = \mathbb{E}[B(t)]\,\mathbb{E}[W^2(t)]$. But since $\mathbb{E}[B(t)] = 0$ we would require $\mathbb{E}[B(t)W^2(t)] = 0$. We have shown above that $\mathbb{E}[B(t)W^2(t)] > 0$ and thus $B(t)$ and $W(t)$ are not independent.

Exercise 4.20 (Local time)

(i)

$$f'(x) = \begin{cases} 1 & \text{if } x > K \\ \text{undefined} & \text{if } x = K \\ 0 & x < K \end{cases}$$

$$f''(x) = \begin{cases} 0 & \text{if } x \neq K \\ \text{undefined} & \text{if } x = K \end{cases}$$

(ii)

$$f(W(T)) = f(W(0)) + \int_0^T \mathbb{I}_{\{W(t)>K\}} dW(t) \tag{9}$$

Using that by Theorem 3.3.2(i), $W(T) \sim \mathcal{N}(0,T)$, we can compute the expected value of the right hand side as

$$
\begin{aligned}
\mathbb{E}\left[f(W(T))\right] &= \mathbb{E}\left[(W(T)-K)^+\right] \\
&= \int_{-\infty}^{\infty} \left(\sqrt{T}x - K\right)^+ \mathcal{N}'(x)dx \\
&= \int_{\frac{K}{\sqrt{T}}}^{\infty} \left(\sqrt{T}x - K\right) \mathcal{N}'(x)dx \\
&= \sqrt{T}\int_{\frac{K}{\sqrt{T}}}^{\infty} x\mathcal{N}'(x)dx - K\mathcal{N}\left(-\frac{K}{\sqrt{T}}\right) \\
&= \sqrt{\frac{T}{2\pi}}\int_{\frac{K}{\sqrt{T}}}^{\infty} xe^{-\frac{x^2}{2}}dx - K\mathcal{N}\left(-\frac{K}{\sqrt{T}}\right) \\
&= -\sqrt{\frac{T}{2\pi}}e^{-\frac{x^2}{2}}\Bigg|_{\frac{K}{\sqrt{T}}}^{\infty} - K\mathcal{N}\left(-\frac{K}{\sqrt{T}}\right) \\
&= \sqrt{\frac{T}{2\pi}}e^{-\frac{K^2}{2T}} - K\mathcal{N}\left(-\frac{K}{\sqrt{T}}\right)
\end{aligned}
$$

Since the Itô integral is a martingale starting at zero, it has zero expected value and we get for the right hand side

$$
\begin{aligned}
\mathbb{E}\left[f(W(0))\right] + \mathbb{E}\left[\int_0^T \mathbb{I}_{\{W(t)>K\}}dW(t)\right] &= \mathbb{E}\left[(W(0)-K)^+\right] \\
&= (0-K)^+ \\
&= 0.
\end{aligned}
$$

This shows that Equation (9) does not hold.

(iii) The answer to this question is pretty much given in the hints already.

$$
f'(x) = \begin{cases} 0 & \text{if } x \leq K - \frac{1}{2n} \\ n(x-K) + \frac{1}{2} & \text{if } K - \frac{1}{2n} \leq x \leq K + \frac{1}{2n} \\ 1 & \text{if } x \geq K + \frac{1}{2n} \end{cases}
$$

$$
f''(x) = \begin{cases} 0 & \text{if } x < K - \frac{1}{2n} \\ n & \text{if } K - \frac{1}{2n} < x < K + \frac{1}{2n} \\ 0 & \text{if } x > K + \frac{1}{2n} \end{cases}
$$

(iv) Similar to (iii), there is not too much to add to the hints. We first note that the intervals of the piecewise function definition have limits

$$\begin{aligned}
\lim_{n\to\infty}\left[0, K-\frac{1}{2n}\right] &= [0,K) \\
\lim_{n\to\infty}\left[K-\frac{1}{2n}, K+\frac{1}{2n}\right] &= K \\
\lim_{n\to\infty}\left[K+\frac{1}{2n}, \infty\right) &= (K,\infty).
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \begin{cases} 0 & \text{if } x \leq K \\ x-K & \text{if } x \geq K \end{cases} \\
&= (x-K)^+ \\
\lim_{n\to\infty} f_n'(x) &= \begin{cases} 0 & \text{if } x < K \\ \frac{1}{2} & \text{if } x = K \\ 1 & \text{if } x > K \end{cases} \cdot
\end{aligned} \tag{10}$$

(v) Let $M(T) = \max_{0\leq t\leq T} W(T)$ be the maximum of a fixed Brownian motion path ω during the time interval $[0,T]$. By assumption, $M(T) < K$ and by definition

$$L_K(T) = \lim_{n\to\infty} n\int_0^T \mathbb{I}_{\left\{W(t)\in\left(K-\frac{1}{2n},K+\frac{1}{2n}\right)\right\}} dt.$$

Now, from $M(T) < K$ it follows that there exists an n_0 such that $K - \frac{1}{2n} > M(T)$ for all $n \geq n_0$. Consequently, for all $n \geq n_0$, the indicator function inside the integral evaluates to zero for all $t \in [0,T]$ and we get $L_K(T) = 0$.

(vi) Taking expectation on both sides and using that by Theorem 4.3.1(iv) the Itô integral is a martingale with zero expected value yields

$$\mathbb{E}\left[L_K(T)\right] = \mathbb{E}\left[(W(T)-K)^+\right] > 0,$$

where the inequality follows from our result in (ii). Although it is not immediately obvious from the final result, it is enough to note that by the normal distribution of $W(T)$ values of $W(T) > K$ have a strictly positive probability for all finite levels of K and thus the expectation is strictly positive as well. Now,

$$\mathbb{P}\left\{L_K(T) = 0\right\} = 1 \quad \Rightarrow \quad \mathbb{E}\left[L_K(T) = 0\right]$$

and just showed that the implication does not hold. Thus, by contradiction we cannot have that $\mathbb{P}\left\{L_K(T) = 0\right\} = 1$.

Exercise 4.21 (Stop-loss start-gain paradox)

(i) The stop-loss start-gain strategy involves holding a long position in one share whenever the stock price strictly exceeds the strike price and holding no position otherwise. The crucial point is, that buys and sells are not made at the same price and thus the hedger incurs a cost every time he enters and unwinds the stock position.

To make it clear, assume that we by one stock whenever the price level $K + \epsilon$ is crossed from below (and we had no position before) and sell one stock whenever the price level K is crossed from above (and we were long one stock before). Building up and subsequently unwinding the stock position thus yields to a loss of ϵ. Since the mathematical model of a geometric Brownian motion is a process with continuous sample paths, ϵ can theoretically made arbitrarily small. However, smaller values of ϵ lead to more frequent transactions.

Real world stock prices however don't take values in $\mathbb{R}$ but are usually quoted in full USD-cents. Furthermore, they don't move continuously but jump. Thus, each time the stock price crosses the strike price the hedger encounters a friction of at least one USD-cent. Furthermore, the strategy ignores transaction cost that can become very significant for high number of buys and sells in the stock.

(ii) Taking expectation and using that by Theorem 4.3.1(i) the Itô integral is a martingale with zero expected value yields $\mathbb{E}\left[X(T)\right] = 0$. However,

$$
\begin{aligned}
\mathbb{E}\left[(S(T)-K)^+\right] &= \int_{-\infty}^{\infty}\left(S(0)\exp\left\{-\frac{1}{2}\sigma^2 T+\sigma\sqrt{T}x\right\}-K\right)^+\mathcal{N}'(x)dx \\
&= \int_{-\frac{\ln\left(\frac{S(0)}{K}\right)-\frac{1}{2}\sigma^2}{\sigma\sqrt{T}}}^{\infty}\left(S(0)\exp\left\{-\frac{1}{2}\sigma^2 T+\sigma\sqrt{T}x\right\}-K\right)\mathcal{N}'(x)dx \\
&> 0.
\end{aligned}
$$

From the second equation and the properties of the normal distribution it is obvious that the integrand is non-negative everywhere and that it takes strictly positive values for some values of x. Thus, the expected value is strictly positive as well. In conclusion

$$
\mathbb{P}\left(X(T)=(S(T)-K)^+\right)<1,
$$

i.e. there are some paths where the stop-loss start-gain strategy does replicate the option payoff (i.e. all paths where the maximum stock price over the interval $[0,T]$ is strictly less than K). However, since the expected value of the payoff of the strategy does not agree with the expected value of the option payoff, we do not have $X(T)=(S(T)-K)^+$ almost surely. This also follows from the result on the local time in Exercise 4.21. Since a Brownian motion spends a non-zero time at any level with positive probability, so does it at the strike K. But once the strike is crossed, the hedger incurs a cost and the exact replication fails.