Stochastic Calculus for Finance II - some Solutions to Chapter VI

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Exercise 6.1

(i) Let

$$A(u) = \int_t^u \sigma(v)dW(v) + \int_t^u \left(b(v) - \frac{1}{2}\sigma^2(v)\right) dv$$

such that $Z(u) = \exp\{A(u)\}$. For $u = t$, both integrals evaluate to zero and thus $A(t) = 0$ and $Z(t) = 1$. Let $f(u, x) = e^x$ with

$$\frac{\partial f}{\partial u} = 0, \quad \frac{\partial f}{\partial x} = e^x, \quad \frac{\partial^2 f}{\partial x^2} = e^x.$$

Applying Itô's lemma yields for $u \geq t$

$$\begin{aligned} dZ(u) &= df(u, A(t)) \\ &= Z(u)dA(u) + \frac{1}{2}Z(u)dA(u)dA(u) \\ &= \left(b(u) - \frac{1}{2}\sigma^2(u)\right) Z(u)du + \sigma(u)Z(u)dW(u) + \frac{1}{2}\sigma^2(u)Z(u)du \\ &= b(u)Z(u)du + \sigma(u)Z(u)dW(u) \qquad \text{(q.e.d.)}. \end{aligned}$$

(ii) By the Itô product rule, we have

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$$
\begin{aligned}
dX(u) &= d\left(Y(u)Z(u)\right) \\
&= Y(u)dZ(u) + Z(u)dY(u) + dY(u)dZ(u) \\
&= b(u)X(u)du + \sigma(u)X(u)dW(u) + \left(a(u) - \sigma(u)\gamma(u)\right)du + \gamma(u)dW(u) + \sigma(u)\gamma(u)du \\
&= \left(a(u) + b(u)X(u)\right)du + \left(\gamma(u) + \sigma(u)X(u)\right)dW(u) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

Finally, using the previous result for $Y(t)$ and $Z(t)$ we see that $X(t) = Y(t)Z(t) = x$.

Exercise 6.2 (No-Arbitrage Derivation of Bond-Pricing Equation)

(i) The self-financing portfolio process $X(t)$ is given by

$$
\begin{aligned}
dX(t) &= \Delta_1(t)df\left(t, R(t), T_1\right) + \Delta_2(t)df\left(t, R(t), T_2\right) \\
&\quad + R(t)\left(X(t) - \Delta_1(t)f\left(t, R(t), T_1\right) - \Delta_2(t)f\left(t, R(t), T_2\right)\right)dt.
\end{aligned}
$$

Thus, by the Itô product rule,

$$
\begin{aligned}
d(D(t)X(t)) &= D(t)dX(t) + X(t)dD(t) + dD(t)dX(t) \\
&= \Delta_1(t)D(t)df\left(t, R(t), T_1\right) + \Delta_2(t)D(t)df\left(t, R(t), T_2\right) \\
&\quad + R(t)D(t)\left(X(t) - \Delta_1(t)f\left(t, R(t), T_1\right) - \Delta_2(t)f\left(t, R(t), T_2\right)\right)dt \\
&\quad - R(t)D(t)X(t)dt \\
&= \Delta_1(t)D(t)\left[-R(t)f\left(t, R(t), T_1\right)dt + df\left(t, R(t), T_1\right)\right] \\
&\quad + \Delta_2(t)D(t)\left[-R(t)f\left(t, R(t), T_2\right)dt + df\left(t, R(t), T_2\right)\right].
\end{aligned}
$$

Here, we used that

$$
dD(t) = -R(t)D(t)dt
$$

has zero quadratic variation such that the cross-variation term in the Itô differential drops out. Assuming that the zero-coupon bond price is smooth enough such that the Itô formula can be applied, then its differential is given by

$$
\begin{aligned}
df(t,R(t),T) &= f_t(t,R(t),T)dt + f_r(t,R(t),T)dR(t) + \frac{1}{2}f_{rr}(t,R(t),T)dR(t)dR(t)\\
&= f_t(t,R(t),T)dt + \alpha(t,R(t))f_r(t,R(t),T)dt + \gamma(t,R(t))f_r(t,R(t),T)dW(t)\\
&\quad + \frac{1}{2}\gamma^2(t,R(t))f_{rr}(t,R(t),T)dt.
\end{aligned}
$$

Combining this with the previously obtained dynamics of the discounted portfolio value yields

$$
\begin{aligned}
d(D(t)X(t)) &= \Delta_1(t)D(t)\left[-R(t)f\left(t,R(t),T_1\right) + f_t\left(t,R(t),T_1\right) + \alpha(t,R(t))f_r\left(t,R(t),T_1\right)\right.\\
&\quad \left. + \frac{1}{2}\gamma^2(t,R(t))f_{rr}\left(t,R(t),T_1\right)\right] dt + \Delta_2 D(t)\left[-R(t)f\left(t,R(t),T_2\right)\right.\\
&\quad \left. + f_t\left(t,R(t),T_2\right) + \alpha(t,R(t))f_r\left(t,R(t),T_2\right) + \frac{1}{2}\gamma^2(t,R(t))f_{rr}\left(t,R(t),T_2\right)\right] dt\\
&\quad + D(t)\gamma(t,R(t))\left[\Delta_1(t)f_r\left(t,R(t),T_1\right) + \Delta_2(t)f_r\left(t,R(t),T_2\right)\right] dW(t).
\end{aligned}
$$

This is the first equality in Equation (6.9.4). To get the second equality, we simply substitute $\beta\left(t,R(t),T\right)$ and obtain

$$
\begin{aligned}
d(D(t)X(t)) &= \Delta_1(t)D(t)\left[\alpha(t,R(t)) - \beta\left(t,R(t),T_1\right)\right] f_r\left(t,R(t),T_1\right) dt\\
&\quad + \Delta_2(t)D(t)\left[\alpha(t,R(t) - \beta\left(t,R(t),T_2\right)\right] f_r\left(t,R(t),T_2\right) dt\\
&\quad + D(t)\gamma(t,R(t))\left[\Delta_1(t)f_r\left(t,R(t),T_1\right) + \Delta_2(t)f_r\left(t,R(t),T_2\right)\right] dW(t) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

(ii) We first note that by construction of the trading strategy

$$
\begin{aligned}
&\Delta_1(t)f_r\left(t,R(t),T_1\right) + \Delta_2(t)f_2\left(t,R(t),T_2\right)\\
=\ & S(t)f_r\left(t,R(t),T_1\right)f_r\left(r,R(t),T_2\right) - S(t)f_r\left(t,R(t),T_1\right)f_r\left(r,R(t),T_2\right)\\
=\ & 0.
\end{aligned}
$$

Thus, the diffusion term in the dynamics of $D(t)X(t)$ vanishes and the portfolio is instantaneously risk-free. Furthermore,

$$
\begin{aligned}
&\Delta_1(t)D(t)\left[\alpha(t,R(t)) - \beta\left(t,R(t),T_1\right)\right] f_r\left(t,R(t),T_1\right)\\
&+\Delta_2(t)D(t)\left[\alpha(t,R(t) - \beta\left(t,R(t),T_2\right)\right] f_r\left(t,R(t),T_2\right)\\
=\;& S(t)D(t)\left[\beta\left(t,R(t),T_2\right) - \beta\left(t,R(t),T_1\right)\right] f_r\left(t,R(t),T_1\right) f_r\left(t,R(t),T_2\right)\\
\geq\;& 0.
\end{aligned}
$$

The last inequality follows from the definition of $S(t)$. For no-arbitrage to exist, the discounted wealth process of a risk-free portfolio has to be a martingale and thus, the drift has to vanish. This is only the case if $\beta\left(t,R(t),T_2\right) = \beta\left(t,R(t),T_1\right)$. Since T_1 and T_2 are arbitrary maturities, we conclude that $\beta(t,R(t),T)$ has to be independent of T.

(iii) The discounted portfolio process immediately follows from the result obtained in (i) by setting $T_1 = T$, $\Delta_1(t) = \Delta(t)$ and $\Delta_2(t) = 0$ for all $t \geq 0$. Then

$$
\begin{aligned}
d(D(t)X(t)) \;=\;& \Delta(t)D(t)\left[-R(t)f(t,R(t),T) + f_t(t,R(t),T) + \alpha(t,R(t))f_r(t,R(t),T)\right.\\
&\left.+\frac{1}{2}\gamma^2(t,R(t))f_{rr}(t,R(t),T)\right] dt + D(t)\gamma(t,R(t))\Delta(t)f_r(t,R(t),T)dW(t).
\end{aligned}
$$

If $f_r(t,R(t),T) = 0$, then the diffusion term vanishes. For no-arbitrage to exist, the change in the discounted portfolio value must be zero as well. Otherwise a risk-free profit could be made by taking a long or short position in the risk-free portfolio. Consequently,

$$
-R(t)f(t,R(t),T) + f_t(t,R(t),T) + \frac{1}{2}\gamma^2(t,R(t))f_{rr}(t,R(t),T) = 0 \qquad \text{(q.e.d.)}.
$$

Exercise 6.6 (Moment-Generating Function for Cox-Ingersoll-Ross Process)

(i) Let $f(t,x) = e^{\frac{1}{2}bt}x$ where

$$
\frac{\partial f}{\partial t} = \frac{1}{2}be^{\frac{1}{2}bt}x, \quad \frac{\partial f}{\partial x} = e^{\frac{1}{2}bt}, \quad \frac{\partial^2 f}{\partial x^2} = 0.
$$

When applying Itô's formula to compute the differential of $f(t, X_j(t))$, then the geometric drift drops out and we obtain

$$\begin{aligned} d\left(e^{\frac{1}{2}bt}X_j(t)\right) &= be^{\frac{1}{2}bt}X_j(t) - \frac{1}{2}be^{\frac{1}{2}bt}X_j(t)dt + \frac{1}{2}\sigma e^{\frac{1}{2}bt}dW_j(t) \\ &= \frac{1}{2}\sigma e^{\frac{1}{2}bt}dW_j(t). \end{aligned}$$

Integrating over $[0, t]$ yields

$$e^{\frac{1}{2}bt}X_j(t) = X_j(0) + \frac{1}{2}\sigma \int_0^t e^{\frac{1}{2}bu}dW_j(u)$$

and thus

$$X_j(t) = e^{-\frac{1}{2}bt}\left[X_j(0) + \frac{1}{2}\sigma \int_0^t e^{\frac{1}{2}bu}dW_j(u)\right] \qquad \text{(q.e.d.)}.$$

The expected value is

$$\begin{aligned} \mathbb{E}\left[X_j(t)\right] &= e^{-\frac{1}{2}bt}X_j(0) + \frac{1}{2}\sigma e^{-\frac{1}{2}bt}\mathbb{E}\left[\int_0^t e^{\frac{1}{2}bu}dW_j(u)\right] \\ &= e^{-\frac{1}{2}bt}X_j(0), \end{aligned}$$

where we used that by Theorem 4.3.1, the Itô integral is a martingale starting at zero. The variance is

$$\begin{aligned} \mathrm{Var}\left[X_j(t)\right] &= \mathrm{Var}\left[\frac{1}{2}\sigma e^{-\frac{1}{2}bt}\int_0^t e^{\frac{1}{2}bu}dW_j(u)\right] \\ &= \frac{1}{4}\sigma^2 e^{-bt}\mathrm{Var}\left[\int_0^t e^{\frac{1}{2}bu}dW_j(u)\right] \\ &= \frac{1}{4}\sigma^2 e^{-bt}\mathbb{E}\left[\left(\int_0^t e^{\frac{1}{2}bu}dW_j(u)\right)^2\right] \\ &= \frac{1}{4}\sigma^2 e^{-bt}\int_0^t e^{bu}du \\ &= \frac{1}{4b}\sigma^2 e^{-bt}e^{bu}\Big|_{u=0}^{u=t} \\ &= \frac{1}{4b}\sigma^2\left(1 - e^{-bt}\right). \end{aligned}$$

In the first equality, we used that the term $e^{-\frac{1}{2}bt}X_j(0)$ is a constant and thus does not contribute to the variance of $X_j(t)$. In the third equality we again used that the Itô integral is a martingale starting at zero and fourth equality is a consequence of the Itô isometry (Theorem 4.3.1). Finally, by Theorem 4.4.9, the Itô integral of a deterministic integrand is normally distributed and we conclude that

$$X_j(t) \sim \mathcal{N}\left(e^{-\frac{1}{2}bt}X_j(0), \frac{1}{4b}\sigma^2\left(1-e^{-bt}\right)\right) \qquad \text{(q.e.d.)}.$$

(ii) Let $f(t, x_1, x_2, \ldots, x_d) = \sum_{j=1}^d x_j^2$ where

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x_j} = 2x_j, \quad \frac{\partial^2 f}{\partial x_j^2} = 2, \quad \frac{\partial^2 f}{\partial x_j, \partial x_k} = 0.$$

An application of the multidimensional Itô formula yields

$$\begin{aligned} dR(t) &= df(t, X_1(t), X_2(t), \ldots, X_d(t)) \\ &= \sum_{j=1}^d 2X_j(t)dX_j(t) + \sum_{j=1}^d dX_j(t)dX_j(t) \\ &= \sum_{j=1}^d \left[\left(\frac{1}{4}\sigma^2 - bX_j^2(t)\right)dt + \sigma X_j(t)dW_j(t)\right] \\ &= \left(\frac{1}{4}\sigma^2 d - b\sum_{j=1}^d X_j^2(t)\right)dt + \sigma\sum_{j=1}^d X_j(t)dW_j(t) \\ &= (a - bR(t))dt + \sigma\sum_{j=1}^d X_j(t)dW_j(t). \end{aligned}$$

Next, we show that the process

$$B(t) = \sum_{j=1}^d \int_0^t \frac{X_j(s)}{\sqrt{R(s)}}dW_j(s)$$

is a $\mathbb{P}$-Brownian motion. First observe that by Theorem 4.3.1, each of the Itô integrals in the definition of $B(t)$ starts at zero, is a martingale and has continuous sample paths. As a sum over these Itô integrals, the process $B(t)$ inherits these properties. Furthermore,

$$\begin{aligned} dB(t)dB(t) &= \frac{1}{R(t)}\sum_{j=1}^{d} X_j^2(t)dt \\ &= dt, \end{aligned}$$

where we used that the Brownian motions $W_1(t), W_2(t), \ldots, W_d(t)$ are independent. By Lévy's characterization of the Brownian motion (Theorem 4.6.4), it follows that $B(t)$ is a $\mathbb{P}$-Brownian motion. Consequently,

$$dR(t) = (a - bR(t))dt + \sigma\sqrt{R(t)}dB(t) \qquad \text{(q.e.d.)}.$$

(iii) The random variables $X_j(t)$ and $X_k(t)$ are independent for $j \neq k$ since the only source of randomness in the definition of $X_j(t)$ is $W_j(t)$ for any j and the Brownian motions $W_j(t)$ and $W_k(t)$ are independent for $j \neq k$. The mean and standard deviation are obtained by substituting for $X_j(0)$ in the result obtained in part (i). We get

$$X_j(t) \sim \mathcal{N}\left(\mu(t), v(t)\right),$$

where

$$\mu(t) = e^{-\frac{1}{2}bt}\sqrt{\frac{R(0)}{d}}, \quad v(t) = \frac{1}{4b}\sigma^2\left(1 - e^{-bt}\right) \qquad \text{(q.e.d.)}.$$

(iv) Following the hint and using that $X_j(t)$ is normally distributed, we get

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{uX_j^2(t)\right\}\right]\\
&= \int_{-\infty}^{\infty} \exp\left\{ux^2\right\} \frac{1}{\sqrt{2\pi v(t)}} \exp\left\{-\frac{(x-\mu(t))^2}{2v(t)}\right\} dx\\
&= \int_{-\infty}^{\infty} \exp\left\{ux^2\right\} \frac{1}{\sqrt{2\pi v(t)}} \exp\left\{-\frac{x^2-2x\mu(t)+\mu^2(t)}{2v(t)}\right\} dx\\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t)}} \exp\left\{-\frac{x^2(1-2uv(t))-2x\mu(t)+\mu^2(t)}{2v(t)}\right\} dx\\
&= \exp\left\{-\frac{\mu^2(t)}{2v(t)}\right\}\\
&\quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t)}} \exp\left\{-\frac{x^2(1-2uv(t))-2x\mu(t)\pm\mu^2(t)/(1-2uv(t))^2}{2v(t)}\right\} dx\\
&= \exp\left\{-\frac{\mu^2(t)(1-1/(1-2uv(t)))}{2v(t)}\right\}\\
&\quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t)}} \exp\left\{-\frac{\left(x\sqrt{1-2uv(t)}-\mu(t)/\sqrt{1-2uv(t)}\right)^2}{2v(t)}\right\} dx.\\
&= \exp\left\{\frac{\mu^2(t)u}{1-2uv(t)}\right\}\\
&\quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi v(t)}} \exp\left\{-\frac{(1-2uv(t))\left(x-\mu(t)/(1-2uv(t))\right)^2}{2v(t)}\right\} dx.\\
&= \frac{1}{\sqrt{1-2uv(t)}} \exp\left\{\frac{\mu^2(t)u}{1-2uv(t)}\right\}\\
&\quad \int_{-\infty}^{\infty} \sqrt{\frac{1-2uv(t)}{2\pi v(t)}} \exp\left\{-\frac{\left(x-\mu(t)/(1-2uv(t))\right)^2}{2v(t)/(1-2uv(t))}\right\} dx.
\end{aligned}
$$

The integrand is the density of a normal random variable with distribution

$$
\mathcal{N}\left(\frac{\mu(t)}{1-2uv(t)}, \frac{v(t)}{1-2uv(t)}\right)
$$

and thus the integral evaluates to one. We get

$$
\mathbb{E}\left[\exp\left\{uX_j^2(t)\right\}\right] = \frac{1}{\sqrt{1-2uv(t)}} \exp\left\{\frac{\mu^2(t)u}{1-2uv(t)}\right\} \qquad \text{(q.e.d.)}.
$$

This expression is only well defined if

$$
1-2uv(t) > 0 \qquad \Leftrightarrow \qquad u < \frac{1}{2v(t)}.
$$

(v) We have

$$\begin{aligned}
\mathbb{E}\left[\exp\{uR(t)\}\right] &= \mathbb{E}\left[\exp\left\{u\sum_{i=1}^{d} X_j(t)\right\}\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{d}\exp\{uX_j(t)\}\right] \\
&= \prod_{j=1}^{d}\mathbb{E}\left[\exp\{uX_j(t)\}\right] \\
&= \left(\mathbb{E}\left[\exp\{uX_j(t)\}\right]\right)^d \\
&= \left(\frac{1}{1-2uv(t)}\right)^{d/2}\exp\left\{\frac{d\mu^2(t)u}{1-2uv(t)}\right\},
\end{aligned}$$

where we used that the $X_j(t)$ are independent and identically distributed in the third and fourth equality. Alternatively, we could have directly applied Theorem 2.2.7 which states that the joint moment generating function factors for sums of independent random variables. Now substituting for $\mu(t)$ and d yields

$$\mathbb{E}\left[\exp\{uR(t)\}\right] = \left(\frac{1}{1-2uv(t)}\right)^{2a/\sigma^2}\exp\left\{\frac{e^{-\frac{1}{2}bt}R(0)u}{1-2uv(t)}\right\} \qquad \text{(q.e.d.)}.$$

Exercise 6.8 (Kolmogorov Backward Equation)

Let $h(y)$ be a Borel measurable function and define

$$g(t,x) = \mathbb{E}^{t,x}\left[h(X(T))\middle|\mathcal{F}(s)\right] = \int_0^\infty h(y)p(t,T,x,y)dy.$$

The lower bound at zero is due to the assumption of $X(t)$ being strictly positive. By Lemma 6.4.2, $g(t,X(t))$ is a martingale. Its partial derivatives are

$$\begin{aligned}
g_t(t,x) &= \int_0^\infty h(y)p_t(t,T,x,y)dy \\
g_x(t,x) &= \int_0^\infty h(y)p_x(t,T,x,y)dy \\
g_{xx}(t,x) &= \int_0^\infty h(y)p_{xx}(t,T,x,y)dy
\end{aligned}$$

The differential of $g(t,x)$ is

$$\begin{aligned} dg(t, X(t)) &= g_t(t, X(t))dt + g_x(t, X(t))dX(t) + \frac{1}{2}g_{xx}(t, X(t))(dX(t))^2 \\ &= g_t(t, X(t))dt + \beta(t, X(t))g_x(t, X(t))dt + \gamma(t, X(t))g_x(t, X(t))dW(t) \\ &\quad + \frac{1}{2}\gamma^2(t, X(t))g_{xx}(t, X(t))dt \end{aligned}$$

Since $g(t, X(t))$ is a martingale, the drift term has to be equal to zero

$$g_t(t, X(t)) + \beta(t, X(t))g_x(t, X(t)) + \frac{1}{2}\gamma^2(t, X(t))g_{xx}(t, X(t)) = 0$$

Substituting the derivatives yields

$$0 = \int_0^\infty h(y) \left[p_t(t, T, x, y) + \beta(t, X(t))p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, X(t))p_{xx}(t, T, x, y) \right] dy$$

In order for this equation to hold for all values of $h(y) > 0$, we require

$$-p_t(t, T, x, y) = \beta(t, X(t))p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, X(t))p_{xx}(t, T, x, y) \qquad \text{(q.e.d.)}$$

Exercise 6.10 (Implying the Volatility Surface)

(i) We get

$$\begin{aligned} & -\int_K^\infty (y - K)\frac{\partial}{\partial y}\left(ry\tilde{p}(0, T, x, y)\right) dy \\ &= -(y - K)ry\tilde{p}(0, T, x, y)\big|_{y=K}^{y=\infty} + \int_K^\infty ry\tilde{p}(0, T, x, y)dy \\ &= \int_K^\infty ry\tilde{p}(0, T, x, y)dy \qquad \text{(q.e.d.)}. \end{aligned}$$

Here, we used the assumption given in Equation (6.9.55) in the last step.

(ii) We get

$$
\begin{aligned}
& \frac{1}{2}\int_K^\infty (y-K)\frac{\partial^2}{\partial y^2}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)dy \\
= {} & \frac{1}{2}(y-K)\frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\Big|_{y=K}^{y=\infty} - \frac{1}{2}\int_K^\infty \frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)dy \\
= {} & -\frac{1}{2}\int_K^\infty \frac{\partial}{\partial y}\left(\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)dy \\
= {} & -\frac{1}{2}\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\Big|_{y=K}^{y=\infty} \\
= {} & \frac{1}{2}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

Here, we used the assumption given in Equation (6.9.57) in the second equality and the assumption given in Equation (6.9.60) in the fourth equality.

(iii) Starting from Equation (6.9.53), we get

$$
\begin{aligned}
& c_T(0,T,x,K) \\
= {} & -rc(0,T,x,K) + e^{-rT}\int_K^\infty (y-K)\tilde{p}_T(0,T,x,y)dy \\
= {} & -re^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy \\
& +e^{-rT}\int_K^\infty (y-K)\left(-\frac{\partial}{\partial y}\left(ry\tilde{p}(0,T,x,y)+\frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma^2(T,y)y^2\tilde{p}(0,T,x,y)\right)\right)dy \\
= {} & -re^{-rT}\int_K^\infty (y-K)\tilde{p}(0,T,x,y)dy + re^{-rT}\int_K^\infty y\tilde{p}(0,T,x,y)dy \\
& +\frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K) \\
= {} & rKe^{-rT}\int_K^\infty \tilde{p}(0,T,x,y)dy + \frac{1}{2}e^{-rT}\sigma^2(T,K)K^2\tilde{p}(0,T,x,K).
\end{aligned}
$$

This is the first equality in Equation (6.9.59). Here, we used the Kolmogorov forward equation from Equation (6.9.51) in the second equality and the results from (i) and (ii) in the third. Next, we use the result from Exercise 5.9, namely that the risk-neutral distribution can be represented as

$$
\tilde{p}(0,T,x,K) = e^{rT}c_{KK}(0,T,x,K).
$$

Thus

$$
\begin{aligned}
& c_T(0,T,x,K) \\
= \;& rK\int_K^\infty c_{KK}(0,T,x,y)dy + \frac{1}{2}\sigma^2(T,K)K^2 c_{KK}(0,T,x,K) \\
= \;& rKc_K(0,T,x,y)|_{y=K}^{y=\infty} + \frac{1}{2}\sigma^2(T,K)K^2 c_{KK}(0,T,x,K) \\
= \;& -rKc_K(0,T,x,K) + \frac{1}{2}\sigma^2(T,K)K^2 c_{KK}(0,T,x,K) \qquad \text{(q.e.d.)},
\end{aligned}
$$

where we used that

$$\lim_{K\to\infty} c(0,T,x,K) = 0$$

and thus

$$\lim_{K\to\infty} c_{KK}(0,T,x,K) = 0.$$