Stochastic Calculus for Finance II
-
some Solutions to Chapter VII

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Exercise 7.1 (Black-Scholes-Merton Equation for the up-and-out Call)

(i) We have

$$
\begin{aligned}
\frac{\partial}{\partial t}\delta_p m(\tau,s) &= \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau}\left\{\frac{1}{\sigma\sqrt{\tau}}\left[\ln s + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]\right\} \\
&= -\left\{-\frac{\ln s}{2\sigma\tau\sqrt{\tau}} + \frac{r \pm \frac{1}{2}\sigma^2}{2\sigma\sqrt{\tau}}\right\} \\
&= -\frac{1}{2\sigma\tau\sqrt{\tau}}\left[\ln\left(\frac{1}{s}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right] \\
&= -\frac{1}{2\tau}\delta_\pm\left(\tau, \frac{1}{s}\right) \qquad \text{(q.e.d.)}.
\end{aligned}
$$

(ii) We first compute

$$
\begin{aligned}
\frac{\partial}{\partial s}\delta_p m(\tau,s) &= \frac{\partial}{\partial s}\left\{\frac{1}{\sigma\sqrt{\tau}}\left[\ln s + \left(r \pm \frac{1}{2}\sigma^2\right)\tau\right]\right\} \\
&= \frac{1}{s\sigma\sqrt{\tau}}.
\end{aligned}
$$

Consequently for $s = x/c$, we get

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$$\begin{aligned}\frac{\partial}{\partial x}\delta_{\pm}\left(\tau,\frac{x}{c}\right) &= \frac{\partial}{\partial s}\delta_{\pm}(\tau,s)\frac{\partial s}{\partial x}\\ &= \frac{1}{x\sigma\sqrt{\tau}}\end{aligned}$$

and for $s = x/c$ we have

$$\begin{aligned}\frac{\partial}{\partial x}\delta_{\pm}\left(\tau,\frac{x}{c}\right) &= \frac{\partial}{\partial s}\delta_{\pm}(\tau,s)\frac{\partial s}{\partial x}\\ &= -\frac{1}{x\sigma\sqrt{\tau}} \qquad \text{(q.e.d.)}.\end{aligned}$$

(iii) Since

$$N'\left(\delta_{\pm}(\tau,s)\right) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\delta_{\pm}^2(\tau,s)}{2}\right\},$$

we have

$$\begin{aligned}\frac{N'\left(\delta_{+}(\tau,s)\right)}{N'\left(\delta_{-}(\tau,s)\right)} &= \exp\left\{\frac{1}{2}\left(\delta_{-}^2(\tau,s)-\delta_{+}^2(\tau,s)\right)\right\}\\ &= \exp\left\{\frac{1}{2}\left(\delta_{-}(\tau,s)-\delta_{+}(\tau,s)\right)\left(\delta_{-}(\tau,s)+\delta_{+}(\tau,s)\right)\right\}.\end{aligned}$$

Here,

$$\delta_{-}(\tau,s)-\delta_{+}(\tau,s) = -\sigma\sqrt{\tau}$$

and

$$\delta_{-}(\tau,s)+\delta_{+}(\tau,s) = \frac{2\ln s + 2r\tau}{\sigma\sqrt{\tau}}.$$

Thus,

$$\begin{aligned}\frac{N'\left(\delta_{+}(\tau,s)\right)}{N'\left(\delta_{-}(\tau,s)\right)} &= \exp\left\{-\left(\ln s + r\tau\right)\right\}\\ &= \frac{e^{-r\tau}}{s} \qquad \text{(q.e.d.)}\end{aligned}$$

and

$$e^{-r\tau} N'\left(d_-(\tau, s)\right) = sN'\left(d_+(\tau, s)\right) \qquad \text{(q.e.d.)}.$$

(iv) This result is immediately obvious from the definition of $\delta_\pm(\tau, s)$ and has been used in (iv) already. We have

$$\begin{aligned} \delta_+(\tau, s) - \delta_-(\tau, s) &= \frac{\sigma^2 \tau}{\sigma\sqrt{\tau}} \\ &= \sigma\sqrt{\tau} \qquad \text{(q.e.d.)}. \end{aligned}$$

(v) Again, this result follows immediately from the definition of $\delta_\pm(\tau, s)$. We have

$$\begin{aligned} \delta_\pm(\tau, s) - \delta_\pm\left(\tau, s^{-1}\right) &= \frac{\ln s - \ln s^{-1}}{\sigma\sqrt{\tau}} \\ &= \frac{2\ln s}{\sigma\sqrt{\tau}} \qquad \text{(q.e.d.)}. \end{aligned}$$

(vi) We have

$$\begin{aligned} N''(y) &= \frac{\partial}{\partial y} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \\ &= -\frac{y}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} \\ &= -yN''(y) \qquad \text{(q.e.d.)}. \end{aligned}$$

(vii) To be continued...

Exercise 7.3 (Markov Property for Geometric Brownian Motion and its Maximum to Date)

The crucial steps for the solution to this problem have been derived in Section 7.4.4. First, remember that

$$S(t) = S(0) \exp\left\{\sigma \hat{W}(t)\right\},$$

where $\hat{W}(t)$ is a drifted Brownian motion. The maximum-to-date process is given by

$$Y(t) = S(0)\exp\left\{\sigma\hat{M}(t)\right\},$$

where

$$\hat{M}(t) = \max_{0\le u\le t}\hat{W}(u).$$

First notice that

$$\begin{aligned} S(T) &= S(t)\frac{S(T)}{S(t)} \\ &= S(t)\exp\left\{\sigma\left(\hat{W}(T)-\hat{W}(t)\right)\right\}, \end{aligned}$$

where $S(t)$ is $\mathcal{F}(t)$-measurable and the increment $\hat{W}(T)-\hat{W}(t)$ is independent of the σ-algebra $\mathcal{F}(t)$ since by Definition 3.3.1 Brownian motion has independent increments. Note that adding a drift to the Brownian motion does not change this property. Similarly,

$$\begin{aligned} Y(T) &= Y(t)\frac{Y(T)}{Y(t)} \\ &= Y(t)\exp\left\{\sigma\left(\hat{M}(T)-\hat{M}(t)\right)\right\} \\ &= Y(t)\exp\left\{\sigma\left[\max_{t\le u\le T}\hat{W}(u)-\hat{M}(t)\right]^{+}\right\} \\ &= Y(t)\exp\left\{\sigma\left[\max_{t\le u\le T}\left(\hat{W}(u)-\hat{W}(t)\right)-\left(\hat{M}(t)-\hat{W}(t)\right)\right]^{+}\right\} \\ &= Y(t)\exp\left\{\left[\max_{t\le u\le T}\sigma\left(\hat{W}(u)-\hat{W}(t)\right)-\ln\left(\frac{Y(t)}{S(t)}\right)\right]^{+}\right\}. \end{aligned}$$

Again, $\ln\left(Y(t)/S(t)\right)$ is $\mathcal{F}(t)$-measurable while

$$B(t,T) = \max_{t\le u\le T}\sigma\left(\hat{W}(u)-\hat{W}(t)\right)$$

is independent of the filtration $\mathcal{F}(t)$. Thus,

$$\mathbb{E}\left[f(S(T),Y(T))\middle|\mathcal{F}(t)\right] = \mathbb{E}\left[h\left(S(t),Y(t),\frac{S(T)}{S(t)},B(t,T)\right)\middle|\mathcal{F}(t)\right],$$

where

$$h\left(S(t),Y(t),A(t,T),B(t,T)\right\} = f\left(S(t)A(t,T),Y(t)\exp\left\{\left[B(t,T)-\ln\left(\frac{Y(t)}{S(t)}\right)\right]^{+}\right\}\right).$$

By Lemma 2.3.4, there exists a function $g(S(t),Y(t))$ defined by

$$g(x,y) = \mathbb{E}\left[h\left(x,y,\frac{S(T)}{S(t)},B(t,T)\right)\right]$$

such that

$$\mathbb{E}\left[f(S(T),Y(T))|\,\mathcal{F}(t)\right] = g(S(t),Y(t)) \qquad \text{(q.e.d.)}.$$

Exercise 7.4 (Cross Variation of Geometric Brownian Motion and its Maximum to Date)

We have

$$\begin{aligned}\left|\sum_{j=1}^{m}\left(Y\left(t_j\right)-Y\left(t_{j-1}\right)\right)\left(S\left(t_j\right)-S\left(t_{j-1}\right)\right)\right| &\leq \sum_{j=1}^{m}\left|Y\left(t_j\right)-Y\left(t_{j-1}\right)\right|\left|S\left(t_j\right)-S\left(t_{j-1}\right)\right| \\ &\leq \max_{1\leq j\leq m}\left|S\left(t_j\right)-S\left(t_{j-1}\right)\right|\sum_{j=1}^{m}Y\left(t_j\right)-Y\left(t_{j-1}\right) \\ &= \max_{1\leq j\leq m}\left|S\left(t_j\right)-S\left(t_{j-1}\right)\right|\left(Y(T)-Y(0)\right).\end{aligned}$$

In the limit as maximum step size goes to zero, we get

$$\lim_{||\Pi||\to 0}\max_{1\leq j\leq m}\left|S\left(t_j\right)-S\left(t_{j-1}\right)\right|\left(Y(T)-Y(0)\right) = 0,$$

since the stock price $S(t)$ is a continuous function of time. Consequently,

$$\lim_{||\Pi||\to 0}\sum_{j=1}^{m}\left(Y\left(t_j\right)-Y\left(t_{j-1}\right)\right)\left(S\left(t_j\right)-S\left(t_{j-1}\right)\right) = 0 \qquad \text{(q.e.d.)}.$$

Exercise 7.7 (Zero-Strike Asian Call)

(i) We can split the integral in a $\mathcal{F}(t)$-measurable part and a part independent of $\mathcal{F}(t)$ to get

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\int_0^T S(u)du \middle| \mathcal{F}(t)\right] &= \mathbb{E}^{\mathbb{Q}}\left[\int_0^t S(u)du \middle| \mathcal{F}(t)\right] + \mathbb{E}^{\mathbb{Q}}\left[\int_t^T S(u)du \middle| \mathcal{F}(t)\right] \\
&= \int_0^t S(u)du + \int_t^T \mathbb{E}^{\mathbb{Q}}\left[S(u)\right] du.
\end{aligned}
$$

Using the martingale property of the discounted stock price we can replace $\mathbb{E}^{\mathbb{Q}}\left[S(u)\right]$ with $e^{-r(t-u)}S(t)$ which yields

$$
\begin{aligned}
\ldots &= \int_0^t S(u)du + S(t)\int_t^T e^{-r(t-u)}du \\
&= \int_0^t S(u)du + \frac{S(t)}{r}\left(e^{r(T-t)} - 1\right).
\end{aligned}
$$

It follows that

$$
e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T S(u)du \middle| \mathcal{F}(t)\right] = e^{-r(T-t)}\frac{1}{T}\int_0^t S(u)du + \frac{S(t)}{rT}\left(1 - e^{-r(T-t)}\right)
$$

and

$$
v(t,x,y) = y\frac{e^{-r(T-t)}}{T} + x\frac{1-e^{-r(T-t)}}{rT}.
$$

(ii) The partial derivatives of $v(t,x,y)$ are given by

$$
\frac{\partial v}{\partial t} = (ry - x)\frac{e^{-r(T-t)}}{T}, \quad \frac{\partial v}{\partial x} = \frac{1-e^{-r(T-t)}}{rT}, \quad \frac{\partial v}{\partial y} = \frac{e^{-r(T-t)}}{T}.
$$

Substituting these into the PDE in Equation (7.5.8) yields

$$
(ry - x)\frac{e^{-r(T-t)}}{T} + x\frac{1-e^{-r(T-t)}}{T} + x\frac{e^{-r(T-t)}}{T} = ry\frac{e^{-r(T-t)}}{T} + x\frac{1-e^{-r(T-t)}}{T}.
$$

All terms cancel out and the assertion follows. We further have

$$
\begin{aligned}
v(t,0,y) &= y\frac{e^{-r(T-t)}}{T} \\
v(T,x,y) &= \frac{y}{T}
\end{aligned}
$$

These are just the boundary conditions in Equations (7.5.9) and (7.5.11) for $K = 0$. Note that both expressions are non-negative since y is an integral over the non-negative random variable $S(t)$.

(iii) By the proof of Theorem (7.5.1) and Remark (7.5.2), the hedge ratio is given by the first derivative of the option price w.r.t. to the spot price.

$$\Delta(t) = \frac{1 - e^{-r(T-t)}}{rT}.$$

This quantity is non-random, since it only depends on time but not on the current value of $S(t)$ or its history.

(iv) The differential of the discounted portfolio value is

$$d\left(e^{-rt}X(t)\right) = \Delta(t)e^{-rt}\sigma S(t)dW(t).$$

Using the process for $\Delta(t)$ computed above, we get

$$d\left(e^{-rt}X(t)\right) = \frac{e^{-rt} - e^{-rT}}{rT}\sigma S(t)dW(t).$$

The differential of the option value is

$$\begin{aligned}
dv(t, S(t), Y(t)) &= \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}dS(t) + \frac{\partial v}{\partial y}dY(t) \\
&= (rY(t) - S(t))\frac{e^{-r(T-t)}}{T}dt + \frac{1 - e^{-r(T-t)}}{rT}dS(t) + \frac{e^{-r(T-t)}}{T}S(t)dt \\
&= rY(t)\frac{e^{-r(T-t)}}{T}dt + \frac{1 - e^{-r(T-t)}}{rT}dS(t) \\
&= rv(t, S(t), Y(t))dt + \frac{1 - e^{-r(T-t)}}{rT}\sigma S(t)dW_t.
\end{aligned}$$

By the product rule, we obtain the differential of the discounted option value

$$\begin{aligned}
d\left(e^{-rt}v(t, S(t), Y(t))\right) &= re^{-rt}v(t, S(t), Y(t))dt + e^{-rt}dv(t, S(t), Y(t)) \\
&= \frac{e^{-rt} - e^{-rT}}{rT}\sigma S(t)dW(t).
\end{aligned}$$

Since the differential of the discounted portfolio value and the discounted option value agree, it follows that if we start with a portfolio that has initial value $X(0) = v(0, S(0), Y(0))$ and hold $\Delta(t)$ shares of the asset at each point in time, then the terminal payoffs agree almost surely.