Stochastic Calculus for Finance I
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some Solutions to Chapter II

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Exercise 2.6

Let $1 < s \leq t \leq N$ be two indices in the range $1, \ldots, N$. We need to show that $\mathbb{E}[I_t | \mathcal{F}_s] = I_s$. In the following proof, we will assume that $s < t$ holds, for $s = t$ the identity obviously follows straight away.

$$\begin{aligned}
\mathbb{E}[I_t | \mathcal{F}_s] &= \mathbb{E}\left[\sum_{j=0}^{t-1} \Delta_j (M_{j+1} - M_j) \middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\sum_{j=0}^{s-1} \Delta_j (M_{j+1} - M_j) + \sum_{j=s}^{t-1} \Delta_j (M_{j+1} - M_j) \middle| \mathcal{F}_s\right] \\
&= \sum_{j=0}^{s-1} \Delta_j (M_{j+1} - M_j) + \mathbb{E}\left[\sum_{j=s}^{t-1} \Delta_j (M_{j+1} - M_j)\right] \\
&= I_s + \mathbb{E}\left[\sum_{j=s}^{t-1} \mathbb{E}[\Delta_j (M_{j+1} - M_j) | \mathcal{F}_j]\right] \\
&= I_s + \mathbb{E}\left[\sum_{j=s}^{t-1} \Delta_j \mathbb{E}[(M_{j+1} - M_j) | \mathcal{F}_j]\right] \\
&= I_s + \mathbb{E}\left[\sum_{j=s}^{t-1} \Delta_j \cdot 0\right] \\
&= I_s \quad \text{q.e.d.}
\end{aligned}$$

We have use the following properties in the different steps of this proof

3. the first sum is $\mathcal{F}_s$-measurable, the second sum is independent of $\mathcal{F}_s$

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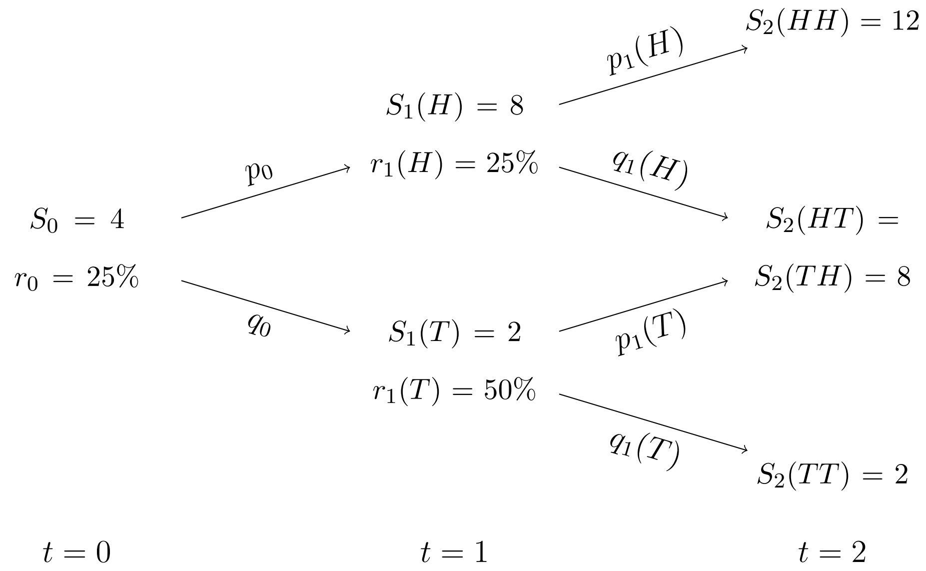
4. tower law of conditional expectation

5. Δ_j is $\mathcal{F}_j$-measurable

6. $\mathbb{E}\left[M_{j+1} - M_j \middle| \mathcal{F}_j\right] = \mathbb{E}\left[M_{j+1} \middle| \mathcal{F}_j\right] - M_j = M_j - M_j = 0$ since M_j is a martingale

Exercise 2.9

The stock-price and interest-rate tree takes the following form:



(i) The tree representation already suggests that the risk-neutral probabilities at $t = 1$ are path-dependent since the interest-rate is stochastic. We thus start by computing $\tilde{p}_0, \tilde{p}_1(H)$ and $\tilde{p}_1(T)$.

$$u_0 = \tfrac{S_1(H)}{S_0} = 2, \quad d_0 = \tfrac{S_1(T)}{S_0} = \tfrac{1}{2}$$
$$\tilde{p}_0 = \tfrac{1+r_0-d_0}{u_0-d_0} = \tfrac{1}{2}, \quad \tilde{q}_0 = 1 - \tilde{p}_0 = \tfrac{1}{2}$$

$$u_1(H) = \tfrac{S_2(HH)}{S_1(H)} = \tfrac{3}{2}, \quad d_1(H) = \tfrac{S_2(HT)}{S_1(H)} = 1$$
$$\tilde{p}_1(H) = \tfrac{1+r_1(H)-d_1(H)}{u_1(H)-d_1(H)} = \tfrac{1}{2}, \quad \tilde{q}_1(H) = 1 - \tilde{p}_1(H) = \tfrac{1}{2}$$

$$u_1(T) = \tfrac{S_2(TH)}{S_1(T)} = 4, \quad d_1(T) = \tfrac{S_2(TT)}{S_1(T)} = 1$$
$$\tilde{p}_1(T) = \tfrac{1+r_1(T)-d_1(T)}{u_1(T)-d_1(T)} = \tfrac{1}{6}, \quad \tilde{q}_1(T) = 1 - \tilde{p}_1(T) = \tfrac{5}{6}$$

We can then compute the risk-neutral ending node probabilities $\tilde{\mathbb{P}}(\omega_1\omega_2)$.

$$\tilde{\mathbb{P}}(HH) = \tilde{p}_0\tilde{p}_1(H) = \tfrac{1}{4}, \quad \tilde{\mathbb{P}}(HT) = \tilde{p}_0\tilde{q}_1(H) = \tfrac{1}{4}$$
$$\tilde{\mathbb{P}}(TH) = \tilde{q}_0\tilde{p}_1(T) = \tfrac{1}{12}, \quad \tilde{\mathbb{P}}(TT) = \tilde{q}_0\tilde{q}_1(T) = \tfrac{5}{12}$$

(ii) The ending-node values $V_2(\omega_1\omega_2) = \max\{S_2(\omega_1\omega_2) - 7, 0\}$ are

$$V_2(HH) = 5, \quad V_2(HT) = V_2(TH) = 1, \quad V_2(TT) = 0$$

We can then compute $V_1(H), V_1(T)$ and V_0 by

$$\begin{aligned}
V_1(H) &= \frac{\tilde{p}_1(H)V_2(HH) + \tilde{q}_1(H)V_2(HT)}{1 + r_1(H)} = \frac{12}{5} = 2.40 \\
V_1(H) &= \frac{\tilde{p}_1(T)V_2(TH) + \tilde{q}_1(T)V_2(TT)}{1 + r_1(T)} = \frac{1}{9} \approx 0.11 \\
V_1(H) &= \frac{\tilde{p}_0V_1(H) + \tilde{q}_0V_1(T)}{1 + r_0} = \frac{226}{225} \approx 1.00
\end{aligned}$$

(iii)

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{103}{270} \approx 38.15\%$$

(iv)

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = 100\%$$