Stochastic Calculus for Finance I
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some Solutions to Chapter III

Matthias Thul*

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Exercise 3.5

(i)

$$Z_2(HH) = \frac{\tilde{\mathbb{P}}(HH)}{\mathbb{P}(HH)} = \frac{9}{16}, \quad Z_2(HT) = \frac{\tilde{\mathbb{P}}(HT)}{\mathbb{P}(HT)} = \frac{9}{8}$$
$$Z_2(TH) = \frac{\tilde{\mathbb{P}}(TH)}{\mathbb{P}(TH)} = \frac{3}{8}, \quad Z_2(TT) = \frac{\tilde{\mathbb{P}}(TT)}{\mathbb{P}(TT)} = \frac{15}{4}$$

(ii) We first need to compute the actual probabilities $p_0, p_1(H)$ and $p_1(T)$ using the definitions of $\mathbb{P}(HH), \mathbb{P}(HT)$ and $\mathbb{P}(TH)$.

$$\mathbb{P}(HH) = p_0 p_1(H) = \frac{4}{9}, \quad \mathbb{P}(HT) = p_0\left(1 - p_1(H)\right) = \frac{2}{9}, \quad \mathbb{P}(TH) = (1 - p_0)p_1(T) = \frac{2}{9}$$
$$\Downarrow$$
$$p_0 = \frac{2}{3}, \quad p_1(H) = \frac{2}{3}, \quad p_1(T) = \frac{2}{3}$$

We can now compute the remaining values of the Radon-Nikodym derivative process.

$$\begin{aligned}
Z_1(H) &= \mathbb{E}\left[Z \middle| w_1 = H\right] = p_1(H)Z_2(HH) + q_1(H)Z_2(HT) = \frac{3}{4} \\
Z_1(H) &= \mathbb{E}\left[Z \middle| w_1 = T\right] = p_1(T)Z_2(TH) + q_1(T)Z_2(TT) = \frac{3}{2} \\
Z_0 &= \mathbb{E}\left[Z\right] = p_0 Z_1(H) + q_0 Z_1(T) = 1 \quad \text{(as by definition)}
\end{aligned}$$

*The author can be contacted via `<<firstname>>.<<lastname>>@gmail.com` and `http://www.matthiasthul.com`.

(iii)

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\begin{aligned}
V_1(H) &= \frac{1+r_0}{Z_1(H)} \mathbb{E}\left[\frac{Z_2 V_2}{(1+r_0)\,(1+r_1(H))} \middle| w_1 = H \right] \\
&= \frac{1}{Z_1(H)\,(1+r_1(H))} \left[p_1(H) Z_2(HH) V_2(HH) + q_1(H) Z_2(HT) V_2(HT) \right] \\
&= \frac{12}{5} = 2.40 \\
V_1(T) &= \frac{1+r_0}{Z_1(T)} \mathbb{E}\left[\frac{Z_2 V_2}{(1+r_0)\,(1+r_1(T))} \middle| w_1 = T \right] \\
&= \frac{1}{Z_1(T)\,(1+r_1(T))} \left[p_1(T) Z_2(TH) V_2(TH) + q_1(T) Z_2(TT) V_2(TT) \right] \\
&= \frac{1}{9} \approx 0.11 \\
V_0 &= \frac{1}{1+r_0} \left[\frac{\mathbb{P}(HH) Z_2(HH) V_2(HH)}{1+r_1(H)} + \frac{\mathbb{P}(HT) Z_2(HT) V_2(HT)}{1+r_1(H)} \right. \\
&\quad \left. + \frac{\mathbb{P}(TH) Z_2(TH) V_2(TH)}{1+r_1(T)} + \frac{\mathbb{P}(TT) Z_2(TT) V_2(TT)}{1+r_1(T)} \right] \\
&= \frac{226}{225} \approx 1.00
\end{aligned}
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As expected, these prices agree with the ones we earlier computed in exercise 1.(ii).