Stochastic Calculus for Finance I
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some Solutions to Chapter V

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Last Update: June 19, 2015

Exercise 5.5

(i) We know that the symmetric random walk M_n is a martingale. Assume that we first reach the level m at time $\tau_m \leq n$. Due to the martingale property and conditional on reaching level m at τ_m, all paths leading to levels $M_n = m - \alpha$ have the same probability as the paths leading to $M_n = m + \alpha$ with $alpha \in \{0, 2, \ldots, n - \tau_m\}$. Obviously, all paths leading to $M_n + \alpha$ also have $\tau_m \leq n$. Thus, the two sets $\{W_m \geq m \wedge W_n = m - \alpha\}$ and $\{\tau_m \leq n \wedge W_n = m - \alpha\}$ are equivalent and have the same probability as $\{W_n = m + \alpha\}$. Now let $b = m - \alpha$, then $\alpha = m - b$ and we get

$$\mathbb{P}\left\{\tau_m \leq n \wedge W_n = b\right\} = \mathbb{P}\left\{W_n = 2m - b\right\}$$

The values of M_n are binomially distributed with

$$\mathbb{P}\left\{M_n = k\right\} = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

where $k \in \{0, 1, \ldots, n\}$. At step n, $M_n = n$ is thus represented by either $k_{M_n=n} = 0$ or $k_{M_n=n} = n$. Similarly, $M_n = -n$ refers to the index $k_{M_n=-n} = n - k_{M_n=n}$. Without loss of generality, we choose $k_{M_n=n} = n$. I.e. k represents the number of up-steps. We then get for k and $n - k$

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$$\begin{aligned} k &= \frac{n+2m-b)}{2} = \frac{n-b}{2}+m \\ n-k &= \frac{n+b}{2}-m \end{aligned}$$

Plugging back into the binomial density and using all previous results yields

$$\mathbb{P}\{W_m \geq m \wedge W_n = b\} = \left(\frac{n!}{\left(\frac{n-b}{2}+m\right)!\left(\frac{n+b}{2}-m\right)!}\right)\left(\frac{1}{2}\right)^n \quad \text{q.e.d.}$$

(ii) If the random walk is asymmetric, then it is no longer a martingale. Conditional on $\tau_m \leq n$, the paths leading to $W_n = m-\alpha$ have probability

$$\mathbb{P}\{\tau_m \leq n \wedge W_n = m-\alpha\} \quad = \quad \mathbb{P}\{W_{\tau_m} = m\}\binom{\hat{n}}{\hat{k}} p^{\hat{k}} q^{(\hat{n}-\hat{k})}$$

Here, $\hat{n} = n - \tau_n$ is the number of steps between the first hitting time τ_m of level m and n. Again, we have used the fact that $\alpha \in \{0, 2, \ldots, n-\tau_m\}$. $\alpha = 0$ is thus equivalent to an equal amount of up- and down-steps between τ_m and n and thus translates into $\hat{k} = \frac{n-\tau_m}{2}$. Similarly, $\alpha = n-\tau_m$ translates into $\hat{k} = 0$. In general, we thus get for the number of up-steps $\hat{k}$ required from M_{τ_m} to $m-\alpha$: $\hat{k} = \frac{n-\tau_m-\alpha}{2}$.

We now analyse the probability of the reflected path. Conditional on $\tau_m \leq n$, the paths leading to $W_n = m+\alpha$ have probability

$$\mathbb{P}\{W_n = m+\alpha\} = \mathbb{P}\{\tau_m \leq n \wedge W_n = m+\alpha\} = \mathbb{P}\{W_{\tau_m} = m\}\binom{\tilde{n}}{\tilde{k}} p^{\hat{k}} q^{(\hat{n}-\hat{k})}$$

Here, $\tilde{n} = \hat{n}$ and by similar arguments as used to construct $\hat{k}$, we find that $\tilde{k} = \frac{n-\tau_m+\alpha}{2}$. We then see that

$$\binom{\hat{n}}{\hat{k}} = \frac{\hat{n}!}{\hat{k}!(\hat{n}-\hat{k})!} = \frac{(n-\tau)!}{\left(\frac{n-\tau_m-\alpha}{2}\right)!\left(\frac{n-\tau_m+\alpha}{2}\right)!} = \frac{\tilde{n}}{(\tilde{n}-\tilde{k})!\tilde{k}!} = \binom{\tilde{n}}{\tilde{k}}$$

and

$$p^{\tilde{k}} q^{(\tilde{n}-\tilde{k})} = p^{\left(\frac{\tilde{n}+\alpha}{2}\right)} q^{\left(\frac{\tilde{n}-\alpha}{2}\right)} = p^{\left(\frac{\tilde{n}-\alpha}{2}\right)} p^{\alpha} q^{\left(\frac{\tilde{n}+\alpha}{2}\right)} q^{-\alpha} = \left(p^{\alpha} q^{-\alpha}\right)\left(p^{\hat{k}} q^{\hat{n}-\hat{k}}\right)$$

Thus

$$\begin{aligned}
\mathbb{P}\{W_n = m+\alpha\} &= \mathbb{P}\{\tau_m \le n \wedge W_n = m+\alpha\} \\
&= \mathbb{P}\{W_{\tau_m} = m\}\binom{\tilde{n}}{\tilde{k}} p^{\hat{k}} q^{(\hat{n}-\hat{k})} \\
&= \mathbb{P}\{W_{\tau_m} = m\}\binom{\hat{n}}{\hat{k}} \left(p^{\alpha} q^{-\alpha}\right)\left(p^{\hat{k}} q^{\hat{n}-\hat{k}}\right) \\
&= \left(p^{\alpha} q^{-\alpha}\right)\mathbb{P}\{\tau_m \le n \wedge W_n = m-\alpha\}
\end{aligned}$$

Substituting $b = m - \alpha$ yields the "modified" reflection principle for a asymmetric random walk

$$\begin{aligned}
\mathbb{P}\{\tau_m \le n \wedge W_n = b\} &= \left(p^{m-b} q^{b-m}\right)\mathbb{P}\{W_n = 2m-b\} \\
&= \left(p^{m-b} q^{b-m}\right)\binom{n}{\frac{n-b}{2}+m} p^{\left(\frac{n-b}{2}+m\right)} q^{\left(\frac{n+b}{2}-m\right)} \\
&= \left(\frac{n!}{\left(\frac{n-b}{2}+m\right)!\left(\frac{n+b}{2}-m\right)!}\right) p^{\left(\frac{n+b}{2}\right)} q^{\left(\frac{n-b}{2}\right)}
\end{aligned}$$

As expected, for $p = q = \frac{1}{2}$ the above equation simplifies to the result from (i).

Exercise 5.9

(i) We first substitute 2^p for $v(s)$ to get

$$s^p = \frac{2}{5}2^p s^p + \frac{2}{5}\left(\frac{1}{2}\right)^p s^p \quad \Leftrightarrow \quad s^p\left(1 - \frac{2}{5}2^p - \frac{2}{5}\left(\frac{1}{2}\right)^p\right) = 0$$

We thus need to solve for

$$1 - \frac{2}{5}2^p - \frac{2}{5}\left(\frac{1}{2}\right)^p = 0$$

We make a substitution and set $x = 2^p$ and multiply by $-\frac{5}{2}x$ to get

$$x^2 - \frac{5}{2}x + 1 = 0 \quad \Rightarrow \quad x \in \left\{2, \frac{1}{2}\right\} \quad \Rightarrow \quad p \in \{1, -1\} \quad \text{q.e.d.}$$

where we have solved the quadratic equation by standard methods.

(ii)

$$\lim_{s\to\infty} v(s) = \lim_{s\to\infty}\left[As + \frac{B}{s}\right] = \begin{cases} 0 & A = 0 \\ \operatorname{sgn}(A)\cdot\infty & \text{otherwise} \end{cases} \qquad \text{q.e.d.}$$

Here, we have used that $\lim_{s\to\infty}\frac{B}{s} = 0$.

(iii) After some simple transformations, we get

$$\frac{B}{s} - (4-s) = 0 \quad \Leftrightarrow \quad (s-2)^2 = 4 - B$$

This equation obviously only has real solutions if $4 - B \geq 0$ or $B \leq 4$ such that the square-root is defined. For $B \leq 4$, the solutions are $s = 2 \pm \sqrt{4-B}$ and especially $s = 2$ if $B = 4$.

(iv) While the lower part of the piecewise defined function $f^l(s) = 4 - s$ is independent of B, the upper part $f^u(s) = \frac{B}{s}$ increases with B. We thus choose $B = 4$, i.e. set it to the max. value it can attain. We furthermore note, that the function $f^u(s) = \frac{4}{s} \geq f^l(s) = 4 - s$ for all $s > 0$ and thus especially for all attainable s. A first idea would thus be to max. the option value by choosing s_B as small as possible. But this does not guarantee that $f^u(s_B) = f^l(s_B)$. We thus choose s_B in accordance with (iii) such that

$$4 - s_B = \frac{4}{s_B} \quad \Leftrightarrow \quad s_B = 2$$

(v) Give $B = 4$, we equate derivatives and solve for s_B to obtain

$$-1 = -\frac{4}{s_B^2} \quad \Leftrightarrow \quad s_B = \pm 2$$

This shows that for $s_B = 2$, both the function values and the derivatives coincide (smooth pasting condition).