Derivation of the Two-Volatility Down-and-Out Put Pricing Formula

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Under the risk-neutral measure $\tilde{\mathbb{P}}$, the general dynamics of two correlated assets that each follow a geometric Brownian motion are given by

$$\begin{aligned}\frac{dS_t^{(K)}}{S_t^{(K)}} &= (r-\delta_K)\,dt + \sigma_K\left(\rho d\tilde{W}_t^{(B)} + \sqrt{1-\rho^2}d\tilde{W}_t^{(K)}\right),\\ \frac{dS_t^{(B)}}{S_t^{(B)}} &= (r-\delta_B)\,dt + \sigma_B d\tilde{W}_t^{(B)}.\end{aligned}$$

An outside barrier option has a vanilla payoff linked to the strike asset $S_t^{(K)}$ conditional on a barrier trigger that is determined by the overall maximum or minimum of the barrier asset $S_t^{(B)}$. By the risk-neutral pricing formula, the current value of the two-asset down-and-out put is given by

$$U_0 = \tilde{\mathbb{E}}\left[e^{-rT}\left(K - S_T^{(K)}\right)^+ \mathbb{I}_{\left\{\min_{0\le t\le T} S_t^{(B)} > B\right\}}\right].$$

We define a new Brownian motion $\hat{W}_t^{(B)}$ by

$$\hat{W}_t^{(B)} = \tilde{W}_t^{(B)} + \alpha t, \qquad \alpha = \frac{r-\delta-\frac{1}{2}\sigma_B^2}{\sigma_B}$$

By Girsanov's theorem, $\hat{W}_t^{(B)}$ is a Brownian motion under the new probability measure $\hat{\mathbb{P}}$ defined by the Nikodým derivative process

$$Z_t = \exp\left\{-\alpha\tilde{W}_t^{(B)} - \frac{1}{2}\alpha^2 t\right\}.$$

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Let $\hat{m}_T^B = \min_{0\le t\le T} \hat{W}_t^{(B)}$ be the minimum of the $\hat{\mathbb{P}}$-Brownian motion over the time interval $[0, T]$. It follows, that

$$\min_{0\le t\le T} S_t^{(B)} = S_0^{(B)} \exp\left\{\sigma_B \hat{m}_T^B\right\}$$

A lower barrier $B < S_0^{(B)}$ has not been triggered if

$$\min_{0\le t\le T} S_t^{(B)} > B \quad \Leftrightarrow \quad \hat{m}_T^B > \frac{1}{\sigma_B} \ln\left(\frac{B}{S_0^{(B)}}\right) = b.$$

The joint density for the minimum $\hat{m}_T^B$ of the Brownian motion and its terminal value $\hat{W}_T^B$ under the measure $\hat{\mathbb{P}}$ can be found in e.g. Karatzas and Shreve (1988) and is given by

$$\hat{f}_{\hat{m}_T^B, \hat{W}_T^B}(m, w) = -\frac{2(2m - w)}{T\sqrt{2\pi T}} \exp\left\{-\frac{(2m - w)^2}{2T}\right\}.$$

Since we need the joint density under the risk-neutral measure $\tilde{\mathbb{P}}$, we apply the change of measure formula to obtain

$$\begin{aligned}
\tilde{\mathbb{P}}\left\{\hat{m}_T^B \le m, \hat{W}_T^{(B)} \le w\right\} &= \hat{\mathbb{E}}\left[\frac{1}{Z_T} \mathbb{I}_{\left\{\hat{m}_T^B \le m, \hat{W}_T^{(B)} \le w\right\}}\right] \\
&= \hat{\mathbb{E}}\left[\exp\left\{\alpha \hat{W}_T^{(B)} - \frac{1}{2}\alpha^2 T\right\} \mathbb{I}_{\left\{\hat{m}_T^B \le m, \hat{W}_T^{(B)} \le w\right\}}\right] \\
&= \int_{-\infty}^{m} \int_{-\infty}^{w} e^{\alpha y - \frac{1}{2}\alpha^2 T} \hat{f}_{\hat{m}_T^B, \hat{W}_T^B}(x, y)\, dy\, dx.
\end{aligned}$$

Differentiating the cumulative distribution function w.r.t. the each of the upper limits of integration yields

$$\begin{aligned}
\tilde{f}_{\hat{m}_T^B, \hat{W}_T^B}(w, m) &= \frac{\partial^2}{\partial m \partial w} \tilde{\mathbb{P}}\left\{\hat{m}_T^B \le m, \hat{W}_T^{(B)} \le w\right\} \\
&= e^{\alpha w - \frac{1}{2}\alpha^2 T} \hat{f}_{\hat{m}_T^B, \hat{W}_T^B}(w, m). \qquad (1)
\end{aligned}$$

Under the risk-neutral measure $\tilde{\mathbb{P}}$, $\hat{W}_t^{(B)}$ is a Brownian motion with non-zero drift and the joint density of its minimum and terminal value is given by Equation (1).

The setup so far has been very general. For the special case of a two-volatility barrier option, we assume that the strike and the barrier asset have the same initial value and drift and we have a perfect correlation of $\rho = 1$. The only difference remains in their diffusion parameters $\sigma_K \neq \sigma_B$. We first note that the solution to the strike asset's SDE in terms of the Brownian motion $\hat{W}_T^{(B)}$ is given by

$$\begin{aligned} S_T^{(K)} &= S_0 \exp\left\{\left(r-\delta-\frac{1}{2}\sigma_K^2\right)T+\sigma_K\tilde{W}_T^{(B)}\right\} \\ &= S_0 \exp\left\{\left(r-\delta-\frac{1}{2}\sigma_K^2-\alpha\sigma_K\right)T+\sigma_K\hat{W}_T^{(B)}\right\}. \end{aligned}$$

We can now solve for the discounted expected payoff under the risk-neutral measure.

$$\begin{aligned} U_0 &= \int_b^\infty \int_b^w e^{-rT}\left(K-S_0\exp\left\{\left(r-\delta-\frac{1}{2}\sigma_K^2-\alpha\sigma_K\right)T+\sigma_K w\right\}\right)^+ \\ &\quad \times\tilde{\mathbb{P}}\left\{\hat{m}_T^B=m,\hat{W}_T^{(B)}=w\right\}dmdw. \end{aligned}$$

The integrand is non-zero if

$$K-S_0\exp\{\dots\}\geq 0 \quad\Leftrightarrow\quad w\leq\frac{\ln\left(\frac{K}{S_0}\right)-\left(r-\delta-\frac{1}{2}\sigma_K^2-\alpha\sigma_K\right)T}{\sigma_K}=k,$$

and we obtain

$$\begin{aligned} &= \int_b^k\int_b^w e^{-rT}\left(K-S_0\exp\{\dots\}\right)\tilde{\mathbb{P}}\{\dots\}\,dmdw \\ &= \int_b^k e^{-rT}\left(K-S_0\exp\{\dots\}\right)\frac{1}{\sqrt{2\pi T}}\exp\left\{\alpha w-\frac{1}{2}\alpha^2T-\frac{1}{2T}(2m-w)^2\right\}\Big|_{m=b}^{m=w}dw \\ &= K\int_b^k\frac{1}{\sqrt{2\pi T}}\exp\left\{-\left(r+\frac{1}{2}\alpha^2\right)T+\alpha w-\frac{1}{2T}w^2\right\}dw \\ &\quad -S_0\int_b^k\frac{1}{\sqrt{2\pi T}}\exp\left\{-\left(\delta+\frac{1}{2}\sigma_K^2+\alpha\sigma_K+\frac{1}{2}\alpha^2\right)T+(\sigma_K+\alpha)w-\frac{1}{2T}w^2\right\}dw \\ &\quad -K\int_b^k\frac{1}{\sqrt{2\pi T}}\exp\left\{-\left(r+\frac{1}{2}\alpha^2\right)T+\alpha w-\frac{1}{2T}(2b-w)^2\right\}dw \\ &\quad +S_0\int_b^k\frac{1}{\sqrt{2\pi T}}\exp\left\{-\left(\delta+\frac{1}{2}\sigma_K^2+\alpha\sigma_K+\frac{1}{2}\alpha^2\right)T+(\sigma_K+\alpha)w-\frac{1}{2T}(2b-w)^2\right\}. \end{aligned}$$

Denote the four lines of the last equality by A, B, C and D. We note, that they all have a similar structure and start by computing a general solution for this integral. Let β and γ be some arbitrary functions that do not depend on w. We then get

$$
\begin{aligned}
&\int_b^k \frac{1}{\sqrt{2\pi T}} \exp\left\{\beta + \gamma w - \frac{1}{2T} w^2\right\} dw \\
= \; & \exp\left\{\beta + \frac{1}{2}\gamma^2 T\right\} \int_b^k \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{(w-\gamma T)^2}{2T}\right\} dw \\
= \; & \exp\left\{\beta + \frac{1}{2}\gamma^2 T\right\} \int_{\frac{b-\gamma T}{\sqrt{T}}}^{\frac{k-\gamma T}{\sqrt{T}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\
= \; & \exp\left\{\beta + \frac{1}{2}\gamma^2 T\right\} \left[\Phi\left(\frac{k-\gamma T}{\sqrt{T}}\right) - \Phi\left(\frac{b-\gamma T}{\sqrt{T}}\right)\right]. \qquad (2)
\end{aligned}
$$

Here, we applied a change of variable from w to z at the third step by setting

$$
z = \frac{w-\gamma T}{\sqrt{T}}, \qquad \frac{dz}{dw} = \frac{1}{\sqrt{T}}.
$$

We now apply the result from Equation (2) successively to the four integrals.

(i)

$$
\beta = -\left(r + \frac{1}{2}\alpha^2\right) T, \qquad \gamma = \alpha
$$

$$
\begin{aligned}
A \; = \; & K e^{-rT} \left[\Phi\left(\frac{k-\alpha T}{\sqrt{T}}\right) - \Phi\left(\frac{b-\alpha T}{\sqrt{T}}\right)\right] \\
= \; & K e^{-rT} \left[\Phi\left(\frac{\ln\left(\frac{K}{S_0}\right) - \left(r - \delta - \frac{1}{2}\sigma_K^2\right) T}{\sigma_K \sqrt{T}}\right) - \Phi\left(\frac{\ln\left(\frac{B}{S_0}\right) - \left(r - \delta - \frac{1}{2}\sigma_B^2\right) T}{\sigma_B \sqrt{T}}\right)\right]
\end{aligned}
$$

(ii)

$$
\beta = -\left(\delta + \frac{1}{2}\sigma_K^2 + \alpha\sigma_K + \frac{1}{2}\alpha^2\right) T, \qquad \gamma = \sigma_K + \alpha
$$

$$
\begin{aligned}
B \; = \; & -S_0 e^{-\delta T} \left[\Phi\left(\frac{k - (\sigma_K + \alpha) T}{\sqrt{T}}\right) - \Phi\left(\frac{b - (\sigma_K + \alpha) T}{\sqrt{T}}\right)\right] \\
= \; & -S_0 e^{-\delta T} \left[\Phi\left(\frac{\ln\left(\frac{K}{S_0}\right) - \left(r - \delta + \frac{1}{2}\sigma_K^2\right) T}{\sigma_K \sqrt{T}}\right) - \right. \\
& \left. -\Phi\left(\frac{\ln\left(\frac{B}{S_0}\right) - \left(r - \delta - \frac{1}{2}\sigma_B^2 + \sigma_K \sigma_B\right) T}{\sigma_B \sqrt{T}}\right)\right]
\end{aligned}
$$

(iii)

$$\beta = -\left(r + \frac{1}{2}\alpha^2\right)T - \frac{2b^2}{T}, \qquad \gamma = \alpha + \frac{2b}{T}$$

$$\begin{aligned} C &= -Ke^{-rT+2\alpha b}\left[\Phi\left(\frac{k-\alpha T-2b}{\sqrt{T}}\right) - \Phi\left(\frac{b-\alpha T-2b}{\sqrt{T}}\right)\right] \\ &= -Ke^{-rT}\left(\frac{B}{S_0}\right)^{\left(\frac{2(r-\delta)}{\sigma_B^2}-1\right)}\left[\Phi\left(\frac{\ln\left(\frac{K}{S_0}\right)-\left(r-\delta-\frac{1}{2}\sigma_K^2\right)T}{\sigma_K\sqrt{T}} - 2\frac{\ln\left(\frac{B}{S_0}\right)}{\sigma_B\sqrt{T}}\right)\right. \\ &\quad \left. -\Phi\left(\frac{\ln\left(\frac{S_0}{B}\right)-\left(r-\delta-\frac{1}{2}\sigma_B^2\right)T}{\sigma_B\sqrt{T}}\right)\right] \end{aligned}$$

(iv)

$$\beta = -\left(\delta + \frac{1}{2}\sigma_K^2 + \alpha\sigma_K + \frac{1}{2}\alpha^2\right)T - \frac{2b^2}{T}, \qquad \gamma = \sigma_K + \alpha + \frac{2b}{T}$$

$$\begin{aligned} D &= S_0e^{-\delta T+2b(\sigma_K+\alpha)}\left[\Phi\left(\frac{k-(\sigma_K+\alpha)T-2b}{\sqrt{T}}\right) - \Phi\left(\frac{b-(\sigma_K+\alpha)T-2b}{\sqrt{T}}\right)\right] \\ &= S_0e^{-\delta T}\left(\frac{B}{S_0}\right)^{\left(\frac{2(r-\delta)}{\sigma_B^2}-1+\frac{2\sigma_K}{\sigma_B}\right)}\left[\Phi\left(\frac{\ln\left(\frac{K}{S_0}\right)-\left(r-\delta+\frac{1}{2}\sigma_K^2\right)T}{\sigma_K\sqrt{T}} - 2\frac{\ln\left(\frac{B}{S_0}\right)}{\sigma_B\sqrt{T}}\right)\right. \\ &\quad \left. -\Phi\left(\frac{\ln\left(\frac{S_0}{B}\right)-\left(r-\delta-\frac{1}{2}\sigma_B^2+\sigma_K\sigma_B\right)T}{\sigma_B\sqrt{T}}\right)\right] \end{aligned}$$

Summing up these four terms yields the two-volatility pricing formula for a down-an-out put option.